Department of Mathematics

SF1626 Calculus in Several Variable
Exam
Tuesday, June 7, 2016

Skrivtid: 08:00-13:00
Allowed aids: none
Examinator: Mats Boij

The exam consists of nine problems each worth a maximum of four points.

Part A of the exam consists of the first three problems. To the score from part A your bonus points are added. The total score from part A can never exeeed 12 points. The bonus points are added automatically and the number of bonus points can be seen from the result page.

The three following problems make up part B and the three last problems part C, which is primarily intended for the higher grades.

The thresholds for the grades will be given by

Grade	A	B	C	D	E	Fx
Total score	27	24	21	18	16	15
Score on part C	6	3	–	–	–	–

In order to acheive full credit on a problem the solution has to be well presented and easy to follow. This means in particular that all notation should be defined, the logical structure clearly described in words or symbols and that the arguments are well motivated and clearly explained. Solutions that suffer seriously regarding this can not acheive more than two points.

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DEL A

1. Let S be the ellipsoid given by the equation $x^2 + 2y^2 + 3z^2 = 5$.
   (a) Determine a normal vector to S at a point (x_0, y_0, z_0) on S. **(2 p)**
   (b) Determine the values of the constant d for which the plane $x + 2y + 6z = d$ is a tangent plane to S. **(2 p)**

2. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = (y + 2z, x + 3z, 2x + 3y)$ and let C be the straight line segment from $(1, 1, 1)$ to $(3, 3, 3)$.
   (a) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ by means of a parametrization of the curve C. **(2 p)**
   (b) Show that $\mathbf{F}$ is conservative and compute the same line integral by means of a potential. **(2 p)**

3. Let $f(x, y) = \sqrt{\cos(y) + \ln(1 + x)}$ for those x and y where this expression is well-defined. Determine the constants a, b and c such that
$$f(x, y) = a + bx + cy + O(x^2 + y^2).$$
   This means that there is a constant M such that
$$|f(x, y) - (a + bx + cy)| \leq M(x^2 + y^2)$$
   for all points (x, y) in a neighbourhood of the origin. **(4 p)**

DEL B

4. A particle is travelling in an orbit described by the parametrization

$$\mathbf{r}(t) = (\cos \pi t + \sin \pi t, \cos \pi t - \sin \pi t, \pi t), \qquad 0 \le t \le 4.$$

   (a) Compute the velocity, $\mathbf{r}'(t)$, and acceleration, $\mathbf{r}''(t)$, of the particle. **(1 p)**
   (b) Show that the velocity and acceleration are perpendicular to each other. **(1 p)**
   (c) Compute the distance travelled by the particle during the interval $0 \le t \le 4$. **(2 p)**

5. The region D in the plane is given by

$$(x^2 + y^2 - x)^2 \le x^2 + y^2$$

   which in polar coordinates corresponds to the inequality $r \le 1 + \cos\theta$. Determine the coordinates of the centre of mass of the region D if its density is constant. **(4 p)**

6. The vector field $\mathbf{F} = (F_1, F_2, F_3)$ is defined in space and satisfies $\mathbf{div}\,\mathbf{F} = x^2 + y^2 + z^2$. In addition, we know that

$$F_3(x, y, z) = y^2 + xz.$$

   The surface S is the upper half of the unit sphere given by $x^2 + y^2 + z^2 = 1$, $z \ge 0$, and its orientation is given by that the normal vector $\mathbf{N} = (x, y, z)$ has a positive direction. Compute the flux of $\mathbf{F}$ through S. **(4 p)**

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DEL C

7. Let $f(x, y, z)$ be a twice continuously differentiable function in the variables x, y and z which only depends on the distance from the origin. That is, $f(x, y, z) = g(r)$ for some twice differentiable function $g(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$.

   The Laplace operator ∇^2 is defined by
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$
   Compute $\nabla^2 f$ expressed in r, the function $g(r)$ and its derivatives, for $r > 0$. **(4 p)**

8. Show that the equation $4x^2+3y^2+\cos(2x^2+y^2) = 1$ only has the solution $(x, y) = (0, 0)$. **(4 p)**

9. Let $u(x, y)$ be a *harmonic function*, i.e., a function satisfying
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \text{för alla } x, y.$$
   (a) Write down a line integral that computes the average $v(r)$ of $u(x, y)$ over the points (x, y) lying on a circle of radius r centred at the origin. **(1 p)**

   (b) It is possible to compute the derivative $v'(r)$ by differentiation within the integral sign. Show that $v(r)$ is a constant function by applying the divergence theorem on the line integral for $v'(r)$. **(2 p)**

   (c) Use the above to conclude that $v(r) = u(0, 0)$ for all $r > 0$. **(1 p)**