Distributative lattices

Def: A lattice is <u>distributative</u> if for all $x, y, z \in L$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad (*)$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

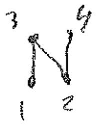
- Examples: B_n "de Morgan's laws"
  D_n since max and min are distributative.

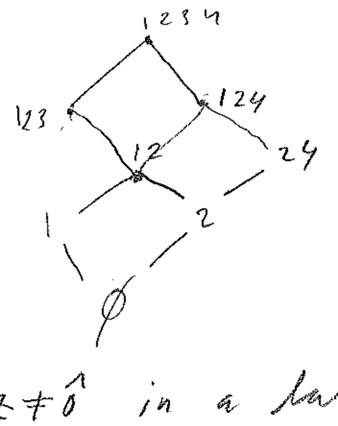
- Let P be a finite poset. An order ideal in P is a set $I \subseteq P$ s.t
  $$x \leq y \in I \Rightarrow x \in I \quad \text{(closed downwards)}$$

- If I, J are order ideals in P, then so are $I \cap J$ and $I \cup J$.

- Let $J(P)$ be the lattice of all order ideals of P, with partial order given by $\subseteq$.

- Since $\cap$ and $\cup$ are distributative $J(P)$ is a distributative lattice

$P =$ 

$J(P) =$ 

- An element $z \neq \hat{0}$ in a lattice is a <u>join-irreducible</u> if we cannot write
  $$z = x \vee y \quad \text{for some } x < z,\ y < z,$$

- Hence any element $z \neq \hat{0}$ in a finite lattice may be written as a join of join-irreducibles.

- If $x_1, \ldots, x_m \in P$ let
$\langle x_1, \ldots, x_m \rangle = \{ z \in P : z \leq x_i \text{ for some } i \} \in J(P)$
be the order ideal generated by $x_1, \ldots, x_m$
- Note that $\langle x_1, \ldots, x_m \rangle = \langle x_1 \rangle \cup \langle x_2, \ldots, x_m \rangle$
- Hence an order ideal $I \in J(P)$ is join-irreducible iff $I = \langle x \rangle$ for some $x \in P$.
- Note that
$$x \leq y \iff \langle x \rangle \subseteq \langle y \rangle$$

- Two posets are <u>isomorphic</u> written $P \cong Q$ if there is an order preserving bijection $\varphi : P \to Q$, whose inverse is also order preserving i.e., $x \leq_P y \iff \varphi(x) \leq_Q \varphi(y)$.

<u>Fundamental theorem for finite distributive lattices</u> (FTFDL): Let L be a finite distr. lattice. The there is a unique (up to isomorphism) poset P for which $L \cong J(P)$.

<u>Proof</u>: Let $P = \{ x \in L : x \text{ is join-irreducible} \}$, considered as a sub-poset of L. Define $\varphi : J(P) \to L$ by $\varphi(I) = \bigvee_{x \in I} x$. $(\varphi(\emptyset) = \hat{0})$
φ is order preserving. If $\xi \in L \setminus \hat{0}$, let $I_\xi = \{ x \in P : x \leq \xi \} \in J(P)$. Then

$\varphi(I_\xi) = \bigvee_{\substack{x \in P \\ x \leq \xi}} x = \xi$, since any element $\xi \neq \hat{0}$ is either a join-irreducible or a join of join-irreducibles

Hence φ is surjective.

Suppose $\varphi(I) = \varphi(J)$ i.e., $\bigvee_{x \in I} x = \bigvee_{y \in J} y = z$

If $x_0 \in I$, then $(x_0 \le z)$

$$x_0 = z \wedge x_0 = x_0 \wedge \Big(\bigvee_{y \in J} y \Big) = \bigvee_{y \in J} (y \wedge x_0)$$

since x_0 is join-irreducible we must have $x_0 \wedge y = x_0$ for some $y \in J$, i.e., $x_0 \le y$. Then $x_0 \in J$.

Hence $I \subseteq J$ and by symmetry $J \subseteq I$.

$\therefore$ $I = J$.

We have proved that φ is a bijection.

Uniqueness: If $L \cong J(Q)$, then the join-irreducibles of $J(Q)$ is $\{\langle x \rangle : x \in Q\} \cong P$.

$\cong Q$

$\square$

- Note that $I \lessdot J$, where $I, J \in J(P)$ iff $J = I \cup \{x\}$, where x is a minimal element of $P \setminus I$.

- Hence $J(P)$ is graded with rank function $\rho(I) = |I|$.

- If P and Q are posets, the <u>cartesian product</u>, $P \times Q$, is defined on $\{(x_1, x_2) : x_1 \in P, x_2 \in Q\}$ by $(x_1, x_2) \le (y_1, y_2)$ iff $x_1 \le_P y_1$ and $x_2 \le_Q y_2$.

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Prop. Let P be a finite poset and $m > 0$. The following quantities are equal.

(a). Number of order-preserving maps $\sigma : P \to \overline{m} = \{1 < 2 < \dots < m\}$

(b). Number of multichains

$$\hat{0} = \emptyset = I_0 \leq I_1 \leq \dots \leq I_m = P = \hat{1}$$

of length m in $J(P)$.

(c). $|J(P \times (\overline{m-1}))|$

Proof: (a)$\Leftrightarrow$(b): If $\sigma : P \to \overline{m}$ define $I_j = \sigma^{-1}(\overline{j})$.

(b)$\Leftrightarrow$(c): Let $A \subseteq P \times (\overline{m-1})$ and let $A_i = \{x : (x,i) \in A\}$. Then A is an order ideal iff A_i is an order ideal of P for all i and if $i \leq j$, then $A_i \supseteq A_j$. Let $I_i := A_{m-i}$. $\square$

Prop. The following quantities are equal:

(a). Number of surjective order preserving maps $\sigma : P \to \overline{m}$.

(b). Number of chains

$$\hat{0} = I_0 < I_1 < \dots < I_m = \hat{1}$$

in $J(P)$.

Proof. Same proof.

When $m = |P|$, then the quantity counts the number $e(P)$ of linear extensions of P.

Incidence algebras

- A poset P is <u>locally finite</u> if every interval in P has finitely many elements. We will assume P is locally finite.

<u>Example</u>: $\mathbb{N}\times\mathbb{N}$, $\mathbb{Z}$

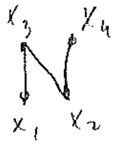
<u>Def.</u> Let K be a field. The incidence algebra $I(P,K)$ (or $I(P)$ if K is understood) is the algebra of all functions $f: \mathrm{Int}(P) \to K$

↑set of all intervals of P.

with multiplication (convolution) given by

$$(fg)(x,y) = \sum_{x \leq z \leq y} f(x,z)\,g(z,y)$$

<u>Remark</u>: If P is finite, then fix a linear extension of P: $x_1, x_2, \ldots, x_p$, (where $x_i <_P x_j \Rightarrow i<j$). Then $I(P,K)$ is isomorphic to the algebra of all upper triangular matrices $A=(A_{ij})_{1\leq i,j\leq p}$ s.t. $A_{ij}=0$ unless $x_i \leq_P x_j$.



$$\begin{bmatrix} * & 0 & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} = A$$

- Note that $I(P)$ has a unique identity element δ defined by $\delta(x,y) = \begin{cases} 1, & x=y \\ 0, & \text{otherwise.} \end{cases}$

- g is a <u>left inverse</u> of f if $gf=\delta$
- —‖— <u>right</u> —‖— $\quad fg=\delta$

Prop. Let $f \in I(P)$. TFAE

a). f has a left inverse

b). —"— right —"—

c). f has a unique two-sided inverse

d). $f(x,x) \neq 0$ for all $x \in P$

Moreover if f^{-1} exists, then $f^{-1}(x,y)$ depends only on the poset $[x,y]$.

Proof: The statement $fg = \delta$ is equivalent to

(1). $f(x,x)\,g(x,x) = 1$ for all $x \in P$, and

(2). $$g(x,y) = -\frac{1}{f(x,x)} \cdot \sum_{x < z \leq y} f(x,z)\,g(z,y)$$

for all $x < y$ in P.

Hence d) is necessary for b) and if b) holds we may compute g recursively b (2). Note that by (2). g only depends on $[x,y]$.

The same reasoning holds for left inverses.

However if $fg_1 = \delta$ and $g_2 f = \delta$, then

$$g_2 = g_2(fg_1) = (g_2 f)g_1 = g_1 .$$

□

The <u>zeta function</u> of P is defined by

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise (of course).} \end{cases}$$

Hence $$\zeta^2(x,y) = \sum_{x \leq z \leq y} 1 \cdot 1 = |[x,y]|$$

- Also, if $m > 0$,

$$\zeta^m(x,y) = \sum_{x=x_0 \le x_1 \le \cdots \le x_m = y} \zeta(x_0,x_1)\zeta(x_1,x_2)\cdots\zeta(x_{m-1},x_m)$$

= number of multichains of length m from x to y.

- Note that

$$(\zeta-\delta)(x,y) = \begin{cases} 1, & \text{if } x<y \\ 0, & \text{otherwise} \end{cases}$$

Hence $(\zeta-\delta)^m(x,y)$ = number of chains $x = x_0 < x_1 < \cdots < x_m = y$ from x to y.