4.4.2 Wandering Domains

Conversely, one can think of the nontransitive map as being obtained from an irrational rotation by "blowing up" some orbits to intervals whose union then makes up the complement of E. These complementary intervals are thus permuted like the points on an orbit for an irrational rotation. All interior points in those intervals are "wandering" in the sense below since they stay within those intervals whose images are all disjoint. The next subsection has an explicit construction of such an example.

Definition 4.4.3 A point is said to be *wandering* if it has a neighborhood all of whose images and preimages are pairwise disjoint.

This behavior is the extreme opposite of recurrence, which we introduce in Definition 6.1.8.

To return to our comparison with the case of rational rotation number we note that in that case a map f is only conjugate to a rotation if all orbits are periodic with the same period and hence $f^q = \mathrm{Id}$ for some $q \in \mathbb{Z}$. Furthermore, a rational rotation can only be a factor when there are infinitely many periodic points, which, as we noted earlier, is unstable.

4.4.3 The Denjoy Example

We now give an example of a nontransitive circle diffeomorphism without periodic points. The construction starts with an irrational rotation and replaces the points of one orbit by suitably chosen intervals. The resulting map is not transitive. This example due to Arnaud Denjoy proves:

Proposition 4.4.4 *For $\rho \in \mathbb{R} \smallsetminus \mathbb{Q}$ there is a nontransitive C^1 diffeomorphism $f\colon S^1 \to S^1$ with $\rho(f) = \rho$.*

Proof If $l_n := (|n| + 3)^{-2}$ and $c_n := 2\left((l_{n+1}/l_n) - 1\right) \geq -1$, then

$$\sum_{n\in\mathbb{Z}} l_n < 2\sum_{n=0}^{\infty} l_n = 2\sum_{n=3}^{\infty} \frac{1}{n^2} < 2\int_2^{\infty} \frac{1}{x^2}\,dx = 1.$$

To "blow up" the orbit $x_n = R_\rho^n x$ of the irrational rotation R_ρ to intervals I_n of length l_n, insert the intervals I_n into S^1 so that they are ordered in the same way as the points x_n and the space between any two such intervals I_m and I_n is

$$\left(1 - \sum_{n\in\mathbb{Z}} l_n\right) d\left(x_m, x_n\right) + \sum_{x_k \in (x_m, x_n)} l_k\,.$$

(This is the sum of the lengths of the intervals I_k inserted in between and the length of the arc of the circle between x_m and x_n, appropriately scaled to reflect the fact that the total length of $S^1 \smallsetminus \bigcup_{n\in\mathbb{Z}} I_n$ is $1 - \sum_{n\in\mathbb{Z}} l_n$.) To define a circle homeomorphism f such that $f(I_n) = I_{n+1}$ and $f\restriction_{S^1 \smallsetminus \bigcup_{n\in\mathbb{Z}} I_n}$ is semiconjugate to a rotation it suffices to specify the derivative $f'(x)$ since f is then obtained by integration.

On the interval $[a, a + l]$ define the tent function

$$h(a, l, x) := 1 - \frac{1}{l}|2(x - a) - l|.$$

Then $h(a, l, a + l/2) = 1$ and $\int_a^{a+l} h(a, l, x)\,dx = l/2$. Denote the left endpoint of I_n by a_n and let

$$f'(x) = \begin{cases} 1 & \text{for } x \in S^1 \setminus \bigcup_{n\in\mathbb{Z}} I_n, \\ 1 + c_n h(a_n, l_n, x) & \text{for } x \in I_n. \end{cases}$$

The choice $c_n = 2\left((l_{n+1}/l_n) - 1\right) = 2\left(l_{n+1} - l_n\right)/l_n$ implies

$$\int_{I_n} f'(x)\,dx = \int_{I_n} (1 + c_n h(a_n, l_n, x))\,dx = l_n + \frac{l_n}{2} c_n = l_{n+1},$$

so indeed $f(I_n) = I_{n+1}$. □

Close inspection of this proof reveals that the derivative of the function f has to be somewhat distorted in order to contract intervals fast enough to fit into the interstices of the universal Cantor set. A systematic careful analysis shows that no sufficiently smooth circle homeomorphism exhibits this phenomenon.

A C^2 diffeomorphism $f\colon S^1 \to S^1$ with irrational rotation number $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$ is transitive and hence topologically conjugate to $R_{\rho(f)}$.

In fact, slightly weaker regularity hypotheses suffice. The most natural weakening is to assume merely that the derivative has bounded variation. A function $g\colon S^1 \to \mathbb{R}$ is said to be of *bounded variation* if its total variation $\mathrm{Var}(g) := \sup \sum_{k=1}^{n} |g(x_k) - g(x_k')|$ is finite. Here the sup is taken over all finite collections $\{x_k, x_k'\}_{k=1}^{n}$ such that x_k, x_k' are endpoints of an interval I_k and $I_k \cap I_j = \varnothing$ for $k \neq j$. Every Lipschitz function and hence every continuously differentiable function has bounded variation.

4.4.4 Dependence of the Rotation Number on a Parameter

Here we examine the dependence of the rotation number on the map as the map is varied. To begin with, it is continuous and monotone.

Proposition 4.4.5 *$\rho(\cdot)$ is continuous in the uniform topology.*

Proof If $\rho(f) = \rho$, take $p'/q', p/q \in \mathbb{Q}$ such that $p'/q' < \rho < p/q$. Pick the lift F of f for which $-1 < F^q(x) - x - p \leq 0$ for some $x \in \mathbb{R}$. Then $F^q(x) < x + p$ for all $x \in \mathbb{R}$, since otherwise $F^q(x) = x + p$ for some $x \in \mathbb{R}$ by the Intermediate-Value Theorem and $\rho = p/q$. Since the function $F^q - \mathrm{Id}$ is periodic and continuous, it attains its maximum. Thus there exists $\delta > 0$ such that $F^q(x) < x + p - \delta$ for all $x \in \mathbb{R}$. This implies that every sufficiently small perturbation $\bar{F}$ of F in the uniform topology also satisfies $\bar{F}^q(x) < x + p$ for all $x \in \mathbb{R}$ and thus $\rho(\bar{F}) < p/q$. A similar argument involving p'/q' completes the proof. □

The definition of the rotation number further suggests that it is monotone: If $F_1 > F_2$, then $\rho(F_1) \geq \rho(F_2)$ follows from the definition. This leads to the following concepts of ordering on the circle and for maps of the circle:

Definition 4.4.6 Define "$<$" on S^1 by $[x] < [y] :\Leftrightarrow y - x \in (0, 1/2) \pmod 1$ and define a partial ordering "$\prec$" on the collection of orientation-preserving circle homeomorphisms by $f_0 \prec f_1 :\Leftrightarrow f_0(x) < f_1(x)$ for all $x \in S^1$.

Notice that neither of these orderings is transitive. Indeed, $[0] < [1/3] < [2/3] < [0]$ and correspondingly $R_0 \prec R_{1/3} \prec R_{2/3} \prec R_0$, where R_α is the