Rigid Body Dynamics (SG2150)
Solution to Exam, 2016-10-27, 8.00-12.00

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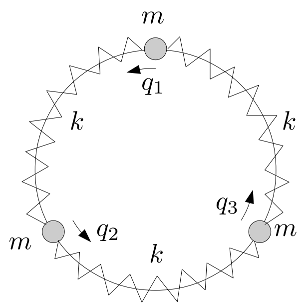
Problem 1.



Use three arc length coordinates q_1, q_2, q_3 for the positions of the masses. The kinetic energy is

$$T = \frac{m}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right).$$

The potential energy is

$$V = \frac{k}{2}\left[\left(q_2 - q_1\right)^2 + \left(q_3 - q_2\right)^2 + \left(q_1 - q_3\right)^2\right].$$

The energies are already quadratic in q, $\dot{q}$, so Lagrange's equations are linear. We find the mass and stiffness matrices to be

$$\boldsymbol{M} = m\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad \boldsymbol{K} = k\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}.$$

The corresponding eigenvalue problem is

$$\left(\boldsymbol{K} - \lambda\boldsymbol{M}\right)\boldsymbol{a} = \boldsymbol{0}.$$

Writing $\lambda = k\mu/m$ we find

$$\det\left(\boldsymbol{K} - \lambda\boldsymbol{M}\right) = -k^3\mu(\mu-3)^2$$

so upon setting the determinant to zero we find the three eigenvalues

$$\lambda_1 = 0, \quad \lambda_{2,3} = 3\frac{k}{m}.$$

The first mode is a drift mode, and after inserting $\lambda = 0$ is the eigenvalue problem we find the shape to be

$$\boldsymbol{a}_1 = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix},$$

which means the system rotates like a rigid body. The second and thirds modes are both oscillatory with angular frequency

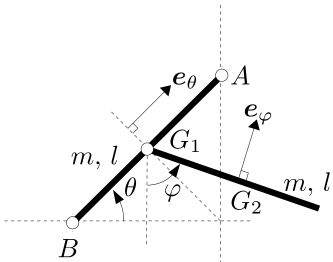
$$\omega_{2,3} = \sqrt{3\frac{k}{m}}.$$

Comments: With a double eigenvalue, there is some freedom in choosing the mode shapes. One possibility is

$$\boldsymbol{a}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{a}_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}.$$

Since the problem is symmetric under rotation by 1/3 turn and under reflection in three planes, we could have predicted three kinds of eigenmodes: One fully symmetric type with single eigenvalues (none of this type here), one type symmetric under rotations but anti-symmetric under reflections, again with single eigenvalues (the first mode is of this type), and finally a type with double eigenvalues and modes (the last two modes here are of this type). The double eigenvalue is thus not an accident.

Problem 2.



Denote the centres of mass of the two rods by G_1 and G_2. Using the "two parts" formula for each rod, we can write the kinetic energy as

$$T = \frac{m}{2}|\boldsymbol{v}_{G_1}|^2 + \frac{ml^2}{12}\frac{\dot\theta^2}{2} + \frac{m}{2}|\boldsymbol{v}_{G_2}|^2 + \frac{ml^2}{12}\frac{\dot\varphi^2}{2}.$$

For the point velocities we find

$$\boldsymbol{v}_{G_1} = \frac{l}{2}\boldsymbol{e}_\theta\dot\theta$$

(the point G_1 is moving on a circle with radius $l/2$) and

$$\boldsymbol{v}_{G_2} = \boldsymbol{v}_{G_1} + \frac{l}{2}\boldsymbol{e}_\varphi\dot\varphi.$$

Using $\boldsymbol{e}_\theta \bullet \boldsymbol{e}_\varphi = \sin(\theta+\varphi)$ we find

$$T = ml^2\left(\frac{7}{24}\dot\theta^2 + \frac{1}{6}\dot\varphi^2 + \frac{1}{4}\sin(\theta+\varphi)\dot\theta\dot\varphi\right).$$

Using the level of the horizontal track as the zero level for the potential energy of gravity, we get

$$V = mg\frac{l}{2}\sin(\theta) + mg\left(\frac{l}{2}\sin(\theta) - \frac{l}{2}\cos(\varphi)\right).$$

The Lagrange function is $L = T - V$ and Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = ml^2\left[\frac{7}{12}\ddot\theta + \frac{1}{4}\sin(\theta+\varphi)\ddot\varphi + \frac{1}{4}\cos(\theta+\varphi)\dot\varphi^2\right] + mgl\cos(\theta) = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) - \frac{\partial L}{\partial\varphi} = ml^2\left[\frac{1}{4}\sin(\theta+\varphi)\ddot\theta + \frac{1}{3}\ddot\varphi + \frac{1}{4}\cos(\theta+\varphi)\dot\theta^2\right] + \frac{mgl}{2}\sin(\varphi) = 0$$

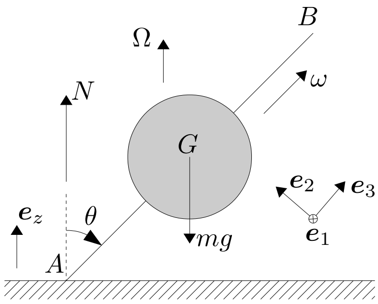
Inserting the initial conditions $\theta(0) = \pi/3$, $\varphi(0) = -\pi/6$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 0$ we get

$$ml^2 \left[\frac{7}{12}\ddot{\theta}(0) + \frac{1}{8}\ddot{\varphi}(0) \right] + \frac{1}{2}mgl = 0$$
$$ml^2 \left[\frac{1}{8}\ddot{\theta}(0) + \frac{1}{3}\ddot{\varphi}(0) \right] - \frac{1}{4}mgl = 0$$

with solution

$$\ddot{\theta}(0) = -\frac{114}{103}\frac{g}{l}$$
$$\ddot{\varphi}(0) = \frac{120}{103}\frac{g}{l}.$$

Problem 3.



Since we are assuming a motion with constant θ, the vertical coordinate of the centre of mass $z_G = l\cos(\theta)$ is also constant. The equation of linear momentum balance $\dot{\boldsymbol{p}} = \boldsymbol{F}$ then gives

$$m\left(\dot{v}_x \boldsymbol{e}_x + \dot{v}_y \boldsymbol{e}_y\right) = (N - mg)\boldsymbol{e}_z$$

where x and y are fixed horizontal directions. Thus

$$N = mg.$$

Now consider angular momentum balance $\dot{\boldsymbol{L}}_G = \boldsymbol{M}_G$. Since the mass distribution is spherical, the moment of inertia operator behaves like a scalar, and we get

$$\dot{\boldsymbol{L}}_G = J\dot{\boldsymbol{\omega}} \text{ with } J = \frac{2}{5}mr^2.$$

Introduce basis vectors $\boldsymbol{e}_3$ along the top axis from A to B, $\boldsymbol{e}_2$ in the top axis vertical plane and $\boldsymbol{e}_1$ horizontal. We have assumed that the angular velocity consist partly of the axis precession, which is also the rotation is the basis triplet $(1,2,3)$

$$\boldsymbol{\omega}^{(1,2,3)} = \Omega \boldsymbol{e}_z$$

and partly of the spin around the axis, so

$$\boldsymbol{\omega} = \boldsymbol{\omega}^{(1,2,3)} + \omega \boldsymbol{e}_3 = \Omega \boldsymbol{e}_z + \omega \boldsymbol{e}_3$$

with ω and Ω both constants. Thus

$$\dot{\boldsymbol{\omega}} = \omega \dot{\boldsymbol{e}}_3 = \omega \left({}^{(1,2,3)}\frac{d\boldsymbol{e}_3}{dt} + \boldsymbol{\omega}^{(1,2,3)} \times \boldsymbol{e}_3 \right) = \omega\Omega \boldsymbol{e}_z \times \boldsymbol{e}_3.$$

Since $\boldsymbol{e}_z = \sin(\theta)\boldsymbol{e}_2 + \cos(\theta)\boldsymbol{e}_3$, we have

$$\dot{\boldsymbol{\omega}} = \omega\Omega\sin(\theta)\boldsymbol{e}_1 \text{ and } \dot{\boldsymbol{L}}_G = J\omega\Omega\sin(\theta)\boldsymbol{e}_1.$$

The torque is

$$\boldsymbol{M}_G = (-l\boldsymbol{e}_3) \times N\boldsymbol{e}_z = mgl\sin(\theta)\boldsymbol{e}_1.$$

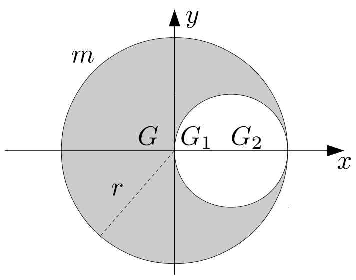
Thus angular momentum is balanced if

$$J\omega\Omega\sin(\theta) = mgl\sin(\theta).$$

If $\theta = 0$, the condition is automatically fulfilled (but neither the $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ directions nor the components Ω or ω are then well defined). When $\theta \neq 0$ we simply get

$$J\omega\Omega = mgl \Rightarrow \omega\Omega = \frac{5}{2}\frac{gl}{r^2}.$$

Problem 4.



We view the big circular plate (subscript $_1$) as the composition of the given body and the removed small circular plate (subscript $_2$). Thus $m_1 = m + m_2$. We also have the ratio $m_1/m_2 = 4$, which gives

$$m_1 = \frac{4}{3}m, \quad m_2 = \frac{1}{3}m.$$

Reflection symmetry in the $y = 0$ plane gives $y_G = 0$. Then, centre of mass for a composite body gives

$$m_1 x_{G_1} = m x_G + m_2 x_{G_2} \Rightarrow \frac{4}{3}m0 = m x_G + \frac{1}{3}m\frac{r}{2} \Rightarrow x_G = -\frac{r}{6}.$$

Since the body is thin, and by the reflection symmetry, the x, y, and z axes are principal axes, so all off-diagonal elements are zero and it is sufficient to compute $J_{G_{xx}}$ and $J_{G_{yy}}$, from which we get $J_{G_{zz}} = J_{G_{xx}} + J_{G_{yy}}$ (again since the body is thin).

Viewed as a composite body, we have

$$J_{1,G_{xx}} = J_{G_{xx}} + J_{2,G_{xx}}$$

where for the large circular plate, and using the parallel axis theorem, we get

$$J_{1,G_{xx}} = J_{1,G_{1xx}} + m_1 0^2 = \frac{1}{4}m_1 r^2 = \frac{1}{3}mr^2.$$

Similarly for the small circular plate we get

$$J_{2,G_{xx}} = J_{2,G_{2xx}} + m_2 0^2 = \frac{1}{4}m_2\left(\frac{r}{2}\right)^2 = \frac{1}{48}mr^2.$$

Thus

$$J_{G_{xx}} = J_{1,G_{xx}} - J_{2,G_{xx}} = \frac{5}{16}mr^2.$$

The y components are in a similar way

$$J_{1,G_{yy}} = J_{1,G_{1yy}} + m_1 \left(\frac{r}{6}\right)^2 = \frac{10}{27} mr^2,$$

$$J_{2,G_{yy}} = J_{2,G_{2yy}} + m_2 \left(\frac{r}{6} + \frac{r}{2}\right)^2 = \frac{73}{432} mr^2,$$

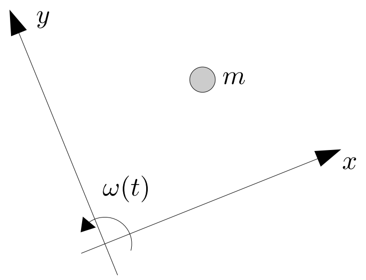
and

$$J_{G_{yy}} = J_{1,G_{yy}} - J_{2,G_{yy}} = \frac{29}{144} mr^2.$$

Finally we have the matrix of all xyz components

$$\boldsymbol{J}_G = \begin{bmatrix} \frac{5}{16} & 0 & 0 \\ 0 & \frac{29}{144} & 0 \\ 0 & 0 & \frac{37}{72} \end{bmatrix} mr^2 = \frac{1}{144} \begin{bmatrix} 45 & 0 & 0 \\ 0 & 29 & 0 \\ 0 & 0 & 74 \end{bmatrix} mr^2.$$

Problem 5.



Writing $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y$ for the position vector of the particle, we get the velocity

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{{}^{(x,y,z)}d\boldsymbol{r}}{dt} + \boldsymbol{\omega}^{(x,y,z)} \times \boldsymbol{r} = (\dot{x} - \omega y)\boldsymbol{e}_x + (\dot{y} + \omega x)\boldsymbol{e}_y.$$

The kinetic energy of the particle is

$$T = \frac{m}{2}\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2\right].$$

There are no external active forces, so $L = T$. Lagrange's equations become

$$\begin{aligned} \frac{d}{dt} m(\dot{x} - \omega y) - m\omega(\dot{y} + \omega x) &= m\left(\ddot{x} - \dot{\omega} y - \omega^2 x - 2\omega \dot{y}\right) = 0 \\ \frac{d}{dt} m(\dot{y} + \omega x) + m\omega(\dot{x} - \omega y) &= m\left(\ddot{y} + \dot{\omega} x - \omega^2 y + 2\omega \dot{x}\right) = 0. \end{aligned}$$

Rearranging into

$$\begin{aligned} m\ddot{x} &= m(\dot{\omega} y + \omega^2 x) + 2m\omega\dot{y} \\ m\ddot{y} &= m(-\dot{\omega} x + \omega^2 y) - 2m\omega\dot{x} \end{aligned}$$

we recognise the first term on the right-hand size as the system point force ($\boldsymbol{a}_{O'}$ is zero), and the second term as the Coriolis force.

Problem 6.

$$\begin{aligned} t_{\text{final}} &= t_{\text{initial}} = t \\ q_{\alpha,\text{final}} &= q_{\alpha,\text{initial}} = q_\alpha \\ \frac{\partial L}{\partial \dot{q}_\alpha}(q, \dot{q}_{\text{final}}, t) &= \frac{\partial L}{\partial \dot{q}_\alpha}(q, \dot{q}_{\text{initial}}, t) + I_\alpha \end{aligned}$$

See "The Theory of Lagrange's Method" section 21.2.