15. Prove that a 3×3 symmetric matrix has only real eigenvalues.
16. A solution $X(t)$ of a system is called *recurrent* if $X(t_n) \to X(0)$ for some sequence $t_n \to \infty$. Prove that a gradient dynamical system has no nonconstant recurrent solutions.
17. Show that a closed bounded omega limit set is connected. Give an example of a planar system having an unbounded omega limit set consisting of two parallel lines.

10 Closed Orbits and Limit Sets

In the past few chapters we have concentrated on equilibrium solutions of systems of differential equations. These are undoubtedly among the most important solutions, but there are other types of solutions that are important as well. In this chapter we will investigate another important type of solution, the *periodic solution* or *closed orbit*. Recall that a periodic solution occurs for $X' = F(X)$ if we have a nonequilibrium point X and a time $\tau > 0$ for which $\phi_\tau(X) = X$. It follows that $\phi_{t+\tau}(X) = \phi_t(X)$ for all t, so ϕ_t is a periodic function. The least such $\tau > 0$ is called the *period* of the solution. As an example, all nonzero solutions of the undamped harmonic oscillator equation are periodic solutions. Like equilibrium points that are asymptotically stable, periodic solutions may also attract other solutions. That is, solutions may limit on periodic solutions just as they can approach equilibria.

In the plane, the limiting behavior of solutions is essentially restricted to equilibria and closed orbits, although there are a few exceptional cases. We will investigate this phenomenon in this chapter in the guise of the important Poincaré-Bendixson theorem. We will see later that, in dimensions greater than two, the limiting behavior of solutions can be quite a bit more complicated.

10.1 Limit Sets

We begin by describing the limiting behavior of solutions of systems of differential equations. Recall that $Y \in \mathbb{R}^n$ is an ω-limit point for the solution through X if there is a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} \phi_{t_n}(X) = Y$. That is, the solution curve through X accumulates on the point Y as time moves forward. The set of all ω-limit points of the solution through X is the ω-limit set of X and is denoted by $\omega(X)$. The α-limit points and the α-limit set $\alpha(X)$ are defined by replacing $t_n \to \infty$ with $t_n \to -\infty$ in the above definition. By a *limit set* we mean a set of the form $\omega(X)$ or $\alpha(X)$.

Here are some examples of limit sets. If X^* is an asymptotically stable equilibrium, it is the ω-limit set of every point in its basin of attraction. Any equilibrium is its own α- and ω-limit set. A periodic solution is the α-limit and ω-limit set of every point on it. Such a solution may also be the ω-limit set of many other points.

Example. Consider the planar system given in polar coordinates by

$$r' = \frac{1}{2}(r - r^3)$$
$$\theta' = 1.$$

As we saw in Section 8.1, all nonzero solutions of this equation tend to the periodic solution that resides on the unit circle in the plane. See Figure 10.1. Consequently, the ω-limit set of any nonzero point is this closed orbit. ■

Example. Consider the system

$$x' = \sin x(-.1 \cos x - \cos y)$$
$$y' = \sin y(\cos x - .1 \cos y).$$

There are equilibria which are saddles at the corners of the square $(0, 0)$, $(0, \pi)$, (π, π), and $(\pi, 0)$, as well as at many other points. There are heteroclinic solutions connecting these equilibria in the order listed. See Figure 10.2. There is also a spiral source at $(\pi/2, \pi/2)$. All solutions emanating from this source accumulate on the four heteroclinic solutions connecting the equilibria (see Exercise 4 at the end of this chapter). Hence the ω-limit set of any point on these solutions is the square bounded by $x = 0, \pi$ and $y = 0, \pi$. ■

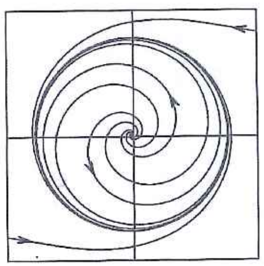

Figure 10.1 The phase plane for $r' = \frac{1}{2}(r - r^3)$, $\theta' = 1$.

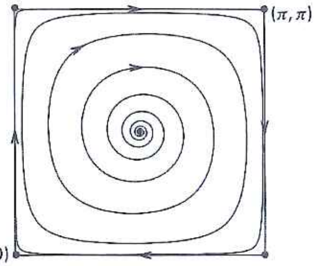

Figure 10.2 The ω-limit set of any solution emanating from the source at $(\pi/2, \pi/2)$ is the square bounded by the four equilibria and the heteroclinic solutions.

In three dimensions there are extremely complicated examples of limit sets, which are not very easy to describe. In the plane, however, limit sets are fairly simple. In fact, Figure 10.2 is typical in that one can show that a closed and bounded limit set other than a closed orbit or equilibrium point is made up of equilibria and solutions joining them. The Poincaré-Bendixson theorem discussed in Section 10.5 states that if a closed and bounded limit set in the plane contains no equilibria, then it must be a closed orbit.

Recall from Section 9.2 that a limit set is closed in $\mathbb{R}^n$ and is invariant under the flow. We shall also need the following result:

Proposition.

1. *If X and Z lie on the same solution curve, then $\omega(X) = \omega(Z)$ and $\alpha(X) = \alpha(Z)$;*
2. *If D is a closed, positively invariant set and $Z \in D$, then $\omega(Z) \subset D$, and similarly for negatively invariant sets and α-limits;*
3. *A closed invariant set, in particular, a limit set, contains the α-limit and ω-limit sets of every point in it.*

Proof: For (1), suppose that $Y \in \omega(X)$, and $\phi_s(X) = Z$. If $\phi_{t_n}(X) \to Y$, then we have

$$\phi_{t_n - s}(Z) = \phi_{t_n}(X) \to Y.$$

Hence $Y \in \omega(Z)$ as well. For (2), if $\phi_{t_n}(Z) \to Y \in \omega(Z)$ as $t_n \to \infty$, then we have $t_n \geq 0$ for sufficiently large n so that $\phi_{t_n}(Z) \in D$. Hence $Y \in D$ since D is a closed set. Finally, part (3) follows immediately from part (2). ■

10.2 Local Sections and Flow Boxes

For the rest of this chapter, we restrict the discussion to planar systems. In this section we describe the local behavior of the flow associated to $X' = F(X)$ near a given point X_0, which is not an equilibrium point. Our goal is to construct first a local section at X_0, and then a flow box neighborhood of X_0. In this flow box, solutions of the system behave particularly simply.

Suppose $F(X_0) \neq 0$. The *transverse line* at X_0, denoted by $\ell(X_0)$, is the straight line through X_0, which is perpendicular to the vector $F(X_0)$ based at X_0. We parametrize $\ell(X_0)$ as follows. Let V_0 be a unit vector based at X_0 and perpendicular to $F(X_0)$. Then define $h : \mathbb{R} \to \ell(X_0)$ by $h(u) = X_0 + uV_0$.

Since $F(X)$ is continuous, the vector field is not tangent to $\ell(X_0)$, at least in some open interval in $\ell(X_0)$ surrounding X_0. We call such an open subinterval containing X_0 a *local section* at X_0. At each point of a local section $\mathcal{S}$, the vector field points "away from" $\mathcal{S}$, so solutions must cut across a local section. In particular $F(X) \neq 0$ for $X \in \mathcal{S}$. See Figure 10.3.

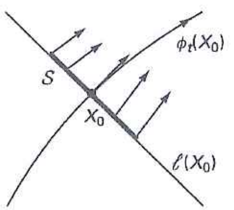

Figure 10.3 A local section $\mathcal{S}$ at X_0 and several representative vectors from the vector field along $\mathcal{S}$.

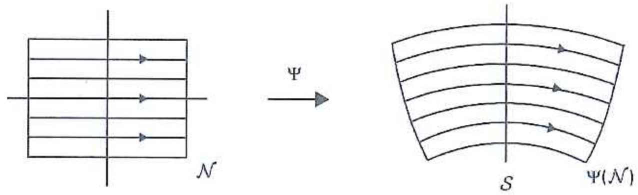

Figure 10.4 The flow box associated to $\mathcal{S}$.

Our first use of a local section at X_0 will be to construct an associated *flow box* in a neighborhood of X_0. A flow box gives a complete description of the behavior of the flow in a neighborhood of a nonequilibrium point by means of a special set of coordinates. An intuitive description of the flow in a flow box is simple: points move in parallel straight lines at constant speed.

Given a local section $\mathcal{S}$ at X_0, we may construct a map Ψ from a neighborhood $\mathcal{N}$ of the origin in $\mathbb{R}^2$ to a neighborhood of X_0 as follows. Given $(s, u) \in \mathbb{R}^2$, we define

$$\Psi(s, u) = \phi_s(h(u))$$

where h is the parameterization of the transverse line described above. Note that Ψ maps the vertical line $(0, u)$ in $\mathcal{N}$ to the local section $\mathcal{S}$; Ψ also maps horizontal lines in $\mathcal{N}$ to pieces of solution curves of the system. Provided that we choose $\mathcal{N}$ sufficiently small, the map Ψ is then one to one on $\mathcal{N}$. Also note that $D\Psi$ takes the constant vector field $(1, 0)$ in $\mathcal{N}$ to vector field $F(X)$. Using the language of Chapter 4, Ψ is a local conjugacy between the flow of this constant vector field and the flow of the nonlinear system.

We usually take $\mathcal{N}$ in the form $\{(s, u) \mid |s| < \sigma\}$ where $\sigma > 0$. In this case we sometimes write $\mathcal{V}_\sigma = \Psi(\mathcal{N})$ and call $\mathcal{V}_\sigma$ the *flow box* at (or about) X_0. See Figure 10.4. An important property of a flow box is that if $X \in \mathcal{V}_\sigma$, then $\phi_t(X) \in \mathcal{S}$ for a unique $t \in (-\sigma, \sigma)$.

If $\mathcal{S}$ is a local section, the solution through a point Z_0 (perhaps far from $\mathcal{S}$) may reach $X_0 \in \mathcal{S}$ at a certain time t_0; see Figure 10.5. We show that in a certain local sense, this "time of first arrival" at $\mathcal{S}$ is a continuous function of Z_0. More precisely:

Proposition. *Let $\mathcal{S}$ be a local section at X_0 and suppose $\phi_{t_0}(Z_0) = X_0$. Let $\mathcal{W}$ be a neighborhood of Z_0. Then there is an open set $\mathcal{U} \subset \mathcal{W}$ containing Z_0 and a differentiable function $\tau : \mathcal{U} \to \mathbb{R}$ such that $\tau(Z_0) = t_0$ and*

$$\phi_{\tau(X)}(X) \in \mathcal{S}$$

for each $X \in \mathcal{U}$.

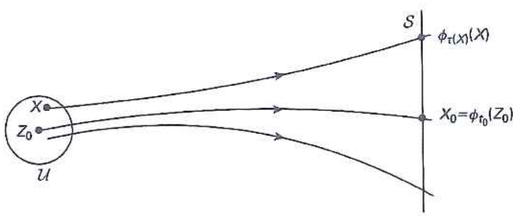

Figure 10.5 Solutions crossing the local section $\mathcal{S}$.

Proof: Suppose $F(X_0)$ is the vector (α, β) and recall that $(\alpha, \beta) \neq (0, 0)$. For $Y = (y_1, y_2) \in \mathbb{R}^2$, define $\eta : \mathbb{R}^2 \to \mathbb{R}$ by

$$\eta(Y) = Y \cdot F(X_0) = \alpha y_1 + \beta y_2.$$

Recall that Y belongs to the transverse line $\ell(X_0)$ if and only if $Y = X_0 + V$ where $V \cdot F(X_0) = 0$. Hence $Y \in \ell(X_0)$ if and only if $\eta(Y) = Y \cdot F(X_0) = X_0 \cdot F(X_0)$.

Now define $G : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ by

$$G(X, t) = \eta(\phi_t(X)) = \phi_t(X) \cdot F(X_0).$$

We have $G(Z_0, t_0) = X_0 \cdot F(X_0)$ since $\phi_{t_0}(Z_0) = X_0$. Furthermore

$$\frac{\partial G}{\partial t}(Z_0, t_0) = |F(X_0)|^2 \neq 0.$$

We may thus apply the implicit function theorem to find a smooth function $\tau : \mathbb{R}^2 \to \mathbb{R}$ defined on a neighborhood $\mathcal{U}_1$ of (Z_0, t_0) such that $\tau(Z_0) = t_0$ and

$$G(X, \tau(X)) \equiv G(Z_0, t_0) = X_0 \cdot F(X_0).$$

Hence $\phi_{\tau(X)}(X)$ belongs to the transverse line $\ell(X_0)$. If $\mathcal{U} \subset \mathcal{U}_1$ is a sufficiently small neighborhood of Z_0, then $\phi_{\tau(X)}(X) \in \mathcal{S}$, as required. ■

10.3 The Poincaré Map

As in the case of equilibrium points, closed orbits may also be stable, asymptotically stable, or unstable. The definitions of these concepts for closed orbits are

entirely analogous to those for equilibria as in Section 8.4. However, determining the stability of closed orbits is much more difficult than the corresponding problem for equilibria. While we do have a tool that resembles the linearization technique that is used to determine the stability of (most) equilibria, generally this tool is much more difficult to use in practice. Here is the tool.

Given a closed orbit γ, there is an associated *Poincaré map* for γ, some examples of which we previously encountered in Sections 1.4 and 6.2. Near a closed orbit, this map is defined as follows. Choose $X_0 \in \gamma$ and let $\mathcal{S}$ be a local section at X_0. We consider the first return map on $\mathcal{S}$. This is the function P that associates to $X \in \mathcal{S}$ the point $P(X) = \phi_t(X) \in \mathcal{S}$ where t is the smallest positive time for which $\phi_t(X) \in \mathcal{S}$. Now P may not be defined at all points on $\mathcal{S}$ as the solutions through certain points in $\mathcal{S}$ may never return to $\mathcal{S}$. But we certainly have $P(X_0) = X_0$, and the previous proposition guarantees that P is defined and continuously differentiable in a neighborhood of X_0.

In the case of planar systems, a local section is a subset of a straight line through X_0, so we may regard this local section as a subset of $\mathbb{R}$ and take $X_0 = 0 \in \mathbb{R}$. Hence the Poincaré map is a real function taking 0 to 0. If $|P'(0)| < 1$, it follows that P assumes the form $P(x) = ax+$ higher order terms, where $|a| < 1$. Hence, for x near 0, $P(x)$ is closer to 0 than x. This means that the solution through the corresponding point in $\mathcal{S}$ moves closer to γ after one passage through the local section. Continuing, we see that each passage through $\mathcal{S}$ brings the solution closer to γ, and so we see that γ is asymptotically stable. We have:

Proposition. *Let $X' = F(X)$ be a planar system and suppose that X_0 lies on a closed orbit γ. Let P be a Poincaré map defined on a neighborhood of X_0 in some local section. If $|P'(X_0)| < 1$, then γ is asymptotically stable.* ■

Example. Consider the planar system given in polar coordinates by

$$r' = r(1 - r)$$
$$\theta' = 1.$$

Clearly, there is a closed orbit lying on the unit circle $r = 1$. This solution in rectangular coordinates is given by $(\cos t, \sin t)$ when the initial condition is $(1, 0)$. Also, there is a local section lying along the positive real axis since $\theta' = 1$. Furthermore, given any $x \in (0, \infty)$, we have $\phi_{2\pi}(x, 0)$, which also lies on the positive real axis $\mathbb{R}^+$. Thus we have a Poincaré map $P: \mathbb{R}^+ \to \mathbb{R}^+$. Moreover, $P(1) = 1$ since the point $x = 1$, $y = 0$ is the initial condition giving the periodic solution. To check the stability of this solution, we need to compute $P'(1)$.

To do this, we compute the solution starting at $(x, 0)$. We have $\theta(t) = t$, so we need to find $r(2\pi)$. To compute $r(t)$, we separate variables to find

$$\int \frac{dr}{r(1-r)} = t + \text{constant}.$$

Evaluating this integral yields

$$r(t) = \frac{xe^t}{1 - x + xe^t}.$$

Hence

$$P(x) = r(2\pi) = \frac{xe^{2\pi}}{1 - x + xe^{2\pi}}.$$

Differentiating, we find $P'(1) = 1/e^{2\pi}$ so that $0 < P'(1) < 1$. Thus the periodic solution is asymptotically stable. ■

The astute reader may have noticed a little scam here. To determine the Poincaré map, we actually first found formulas for all of the solutions starting at $(x, 0)$. So why on earth would we need to compute a Poincaré map? Well, good question. Actually, it is usually very difficult to compute the exact form of a Poincaré map or even its derivative along a closed orbit, since in practice we rarely have a closed-form expression for the closed orbit, never mind the nearby solutions. As we shall see, the Poincaré map is usually more useful when setting up a geometric model of a specific system (see the Lorenz system in Chapter 14). There are some cases where we can circumvent this problem and gain insight into the Poincaré map, as we shall see when we investigate the Van der Pol equation in Section 12.3.

10.4 Monotone Sequences in Planar Dynamical Systems

Let $X_0, X_1, \ldots \in \mathbb{R}^2$ be a finite or infinite sequence of distinct points on the solution curve through X_0. We say that the sequence is *monotone along the solution* if $\phi_{t_n}(X_0) = X_n$ with $0 \le t_1 < t_2 \cdots$.

Let $Y_0, Y_1, \ldots$ be a finite or infinite sequence of points on a line segment I in $\mathbb{R}^2$. We say that this sequence is *monotone along* I if Y_n is between Y_{n-1} and Y_{n+1} in the natural order along I for all $n \ge 1$.

A sequence of points may be on the intersection of a solution curve and a segment I; they may be monotone along the solution curve but not along the segment, or vice versa; see Figure 10.6. However, this is impossible if the segment is a local section in the plane.

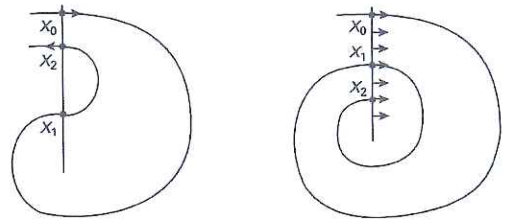

Figure 10.6 Two solutions crossing a straight line. On the left, X_0, X_1, X_2 is monotone along the solution but not along the straight line. On the right, X_0, X_1, X_2 is monotone along both the solution and the line.

Proposition. *Let S be a local section for a planar system of differential equations and let $Y_0, Y_1, Y_2, \ldots$ be a sequence of distinct points in S that lie on the same solution curve. If this sequence is monotone along the solution, then it is also monotone along S.*

Proof: It suffices to consider three points Y_0, Y_1, and Y_2 in S. Let Σ be the simple closed curve made up of the part of the solution between Y_0 and Y_1 and the segment $T \subset S$ between Y_0 and Y_1. Let D be the region bounded by Σ. We suppose that the solution through Y_1 leaves D at Y_1 (see Figure 10.7; if the solution enters D, the argument is similar). Hence the solution leaves D at every point in T since T is part of the local section.

It follows that the complement of D is positively invariant. For no solution can enter D at a point of T; nor can it cross the solution connecting Y_0 and Y_1, by uniqueness of solutions.

Therefore $\phi_t(Y_1) \in \mathbb{R}^2 - D$ for all $t > 0$. In particular, $Y_2 \in S - T$. The set $S - T$ is the union of two half open intervals I_0 and I_1 with Y_j an endpoint of I_j for $j = 0, 1$. One can draw an arc from a point $\phi_\epsilon(Y_1)$ (with $\epsilon > 0$ very small) to a point of I_1, without crossing Σ. Therefore I_1 is outside D. Similarly I_0 is inside D. It follows that $Y_2 \in I_1$ since it must be outside D. This shows that Y_1 is between Y_0 and Y_2 in I, proving the proposition. ■

We now come to an important property of limit points.

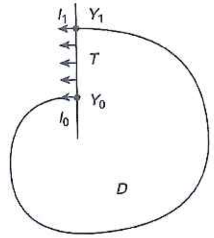

Figure 10.7 Solutions exit the region D through T.

Proposition. *For a planar system, suppose that $Y \in \omega(X)$. Then the solution through Y crosses any local section at no more than one point. The same is true if $Y \in \alpha(X)$.*

Proof: Suppose for the sake of contradiction that Y_1 and Y_2 are distinct points on the solution through Y and $\mathcal{S}$ is a local section containing Y_1 and Y_2. Suppose $Y \in \omega(X)$ (the argument for $\alpha(X)$ is similar). Then $Y_k \in \omega(X)$ for $k = 1, 2$. Let $\mathcal{V}_k$ be flow boxes at Y_k defined by some intervals $J_k \subset \mathcal{S}$; we assume that J_1 and J_2 are disjoint as depicted in Figure 10.8. The solution through X enters each $\mathcal{V}_k$ infinitely often; hence it crosses J_k infinitely often.

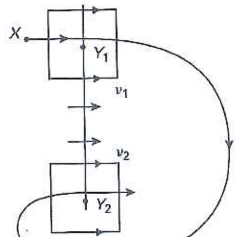

Figure 10.8 The solution through X cannot cross $\mathcal{V}_1$ and $\mathcal{V}_2$ infinitely often.

Hence there is a sequence

$$a_1, b_1, a_2, b_2, a_3, b_3, \ldots,$$

which is monotone along the solution through X, with $a_n \in J_1, b_n \in J_2$ for $n = 1, 2, \ldots$. But such a sequence cannot be monotone along $\mathcal{S}$ since J_1 and J_2 are disjoint, contradicting the previous proposition. ■

10.5 The Poincaré-Bendixson Theorem

In this section we prove a celebrated result concerning planar systems:

Theorem. (Poincaré-Bendixson) *Suppose that Ω is a nonempty, closed and bounded limit set of a planar system of differential equations that contains no equilibrium point. Then Ω is a closed orbit.*

Proof: Suppose that $\omega(X)$ is closed and bounded and that $Y \in \omega(X)$. (The case of α-limit sets is similar.) We show first that Y lies on a closed orbit and later that this closed orbit actually is $\omega(X)$.

Since Y belongs to $\omega(X)$ we know from Section 10.1 that $\omega(Y)$ is a nonempty subset of $\omega(X)$. Let $Z \in \omega(Y)$ and let $\mathcal{S}$ be a local section at Z. Let $\mathcal{V}$ be a flow box associated to $\mathcal{S}$. By the results of the previous section, the solution through Y meets $\mathcal{S}$ in exactly one point. On the other hand, there is a sequence $t_n \to \infty$ such that $\phi_{t_n}(Y) \to Z$; hence infinitely many $\phi_{t_n}(Y)$ belong to $\mathcal{V}$. Therefore we can find $r, s \in \mathbb{R}$ such that $r > s$ and $\phi_r(Y), \phi_s(Y) \in \mathcal{S}$. It follows that $\phi_r(Y) = \phi_s(Y)$; hence $\phi_{r-s}(Y) = Y$ and $r - s > 0$. Since $\omega(X)$ contains no equilibria, Y must lie on a closed orbit.

It remains to prove that if γ is a closed orbit in $\omega(X)$, then $\gamma = \omega(X)$. For this, it is enough to show that

$$\lim_{t\to\infty} d(\phi_t(X), \gamma) = 0,$$

where $d(\phi_t(x), \gamma)$ is the distance from $\phi_t(X)$ to the set γ (that is, the distance from $\phi_t(X)$ to the nearest point of γ).

Let $\mathcal{S}$ be a local section at $Y \in \gamma$. Let $\epsilon > 0$ and consider a flow box $\mathcal{V}_\epsilon$ associated to $\mathcal{S}$. Then there is a sequence $t_0 < t_1 < \cdots$ such that

1. $\phi_{t_n}(X) \in \mathcal{S}$;
2. $\phi_{t_n}(X) \to Y$;
3. $\phi_t(X) \notin \mathcal{S}$ for $t_{n-1} < t < t_n$, $n = 1, 2, \ldots$.

Let $X_n = \phi_{t_n}(X)$. By the first proposition in the previous section, X_n is a monotone sequence in $\mathcal{S}$ that converges to Y.

We claim that there exists an upper bound for the set of positive numbers $t_{n+1} - t_n$ for n sufficiently large. To see this, suppose $\phi_\tau(Y) = Y$ where $\tau > 0$. Then for X_n sufficiently near Y, $\phi_\tau(X_n) \in \mathcal{V}_\epsilon$ and hence

$$\phi_{\tau+t}(X_n) \in S$$

for some $t \in [-\epsilon, \epsilon]$. Thus

$$t_{n+1} - t_n \leq \tau + \epsilon.$$

This provides the upper bound for $t_{n+1} - t_n$. Also, $t_{n+1} - t_n$ is clearly at least 2ϵ, so $t_n \to \infty$ as $n \to \infty$.

Let $\beta > 0$ be small. By continuity of solutions with respect to initial conditions, there exists $\delta > 0$ such that, if $|Z - Y| < \delta$ and $|t| \leq \tau + \epsilon$ then $|\phi_t(Z) - \phi_t(Y)| < \beta$. That is, the distance from the solution $\phi_t(Z)$ to γ is less than β for all t satisfying $|t| \leq \tau + \epsilon$. Let n_0 be so large that $|X_n - Y| < \delta$ for all $n \geq n_0$. Then

$$|\phi_t(X_n) - \phi_t(Y)| < \beta$$

if $|t| \leq \tau + \epsilon$ and $n \geq n_0$. Now let $t \geq t_{n_0}$. Let $n \geq n_0$ be such that

$$t_n \leq t \leq t_{n+1}.$$

Then

$$\begin{aligned} d(\phi_t(X), \gamma) &\leq |\phi_t(X) - \phi_{t-t_n}(Y)| \\ &= |\phi_{t-t_n}(X_n) - \phi_{t-t_n}(Y)| \\ &< \beta \end{aligned}$$

since $|t - t_n| \leq \tau + \epsilon$. This shows that the distance from $\phi_t(X)$ to γ is less than β for all sufficiently large t. This completes the proof of the Poincaré-Bendixson theorem. ■

Example. Another example of an ω-limit set that is neither a closed orbit nor an equilibrium is provided by a *homoclinic solution*. Consider the system

$$\begin{aligned} x' &= -y - \left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}\right)(x^3 - x) \\ y' &= x^3 - x - \left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}\right)y. \end{aligned}$$

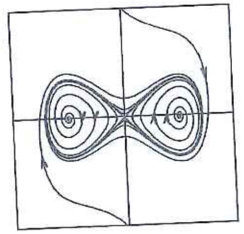

Figure 10.9 A pair of homoclinic solutions in the ω-limit set.

A computation shows that there are three equilibria: at $(0,0)$, $(-1,0)$, and $(1,0)$. The origin is a saddle, while the other two equilibria are sources. The phase portrait of this system is shown in Figure 10.9. Note that solutions far from the origin tend to accumulate on the origin and a pair of homoclinic solutions, each of which leaves and then returns to the origin. Solutions emanating from either source have ω-limit set that consists of just one homoclinic solution and $(0,0)$. See Exercise 6 for proofs of these facts. ■

10.6 Applications of Poincaré-Bendixson

The Poincaré-Bendixson theorem essentially determines all of the possible limiting behaviors of a planar flow. We give a number of corollaries of this important theorem in this section.

A *limit cycle* is a closed orbit γ such that $\gamma \subset \omega(X)$ or $\gamma \subset \alpha(X)$ for some $X \notin \gamma$. In the first case γ is called an ω-limit cycle; in the second case, an α-limit cycle. We deal only with ω-limit sets in this section; the case of α-limit sets is handled by simply reversing time.

In the proof of the Poincaré-Bendixson theorem it was shown that limit cycles have the following property: If γ is an ω-limit cycle, there exists $X \notin \gamma$ such that

$$\lim_{t\to\infty} d(\phi_t(X), \gamma) = 0.$$

Geometrically this means that some solution spirals toward γ as $t \to \infty$. See Figure 10.10. Not all closed orbits have this property. For example, in the

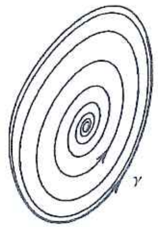

Figure 10.10 A solution spiraling toward a limit cycle.

case of a linear system with a center at the origin in $\mathbb{R}^2$, the closed orbits that surround the origin have no solutions approaching them, and so they are not limit cycles.

Limit cycles possess a kind of (one-sided, at least) stability. Let γ be an ω-limit cycle and suppose $\phi_t(X)$ spirals toward γ as $t \to \infty$. Let S be a local section at $Z \in \gamma$. Then there is an interval $T \subset S$ disjoint from γ, bounded by $\phi_{t_0}(X)$ and $\phi_{t_1}(X)$ with $t_0 < t_1$, and not meeting the solution through X for $t_0 < t < t_1$. See Figure 10.11. The annular region A that is bounded on one side by γ and on the other side by the union of T and the curve

$$\{\phi_t(X) \mid t_0 \le t \le t_1\}$$

is positively invariant, as is the set $B = A - \gamma$. It is easy to see that $\phi_t(Y)$ spirals toward γ for all $Y \in B$. Hence we have:

Corollary 1. *Let γ be an ω-limit cycle. If $\gamma = \omega(X)$ where $X \notin \gamma$, then X has a neighborhood $\mathcal{O}$ such that $\gamma = \omega(Y)$ for all $Y \in \mathcal{O}$. In other words, the set*

$$\{Y \mid \omega(Y) = \gamma\} - \gamma$$

is open. ■

Recall that a subset of R^n that is closed and bounded is said to be *compact*. As another consequence of the Poincaré-Bendixson theorem, suppose that K is a positively invariant set that is compact. If $X \in K$, then $\omega(X)$ must also lie in K. Hence K must contain either an equilibrium point or a limit cycle.

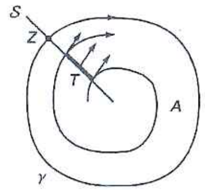

Figure 10.11 The region A is positively invariant.

Corollary 2. *A compact set K that is positively or negatively invariant contains either a limit cycle or an equilibrium point.* ■

The next result exploits the spiraling property of limit cycles.

Corollary 3. *Let γ be a closed orbit and let $\mathcal{U}$ be the open region in the interior of γ. Then $\mathcal{U}$ contains either an equilibrium point or a limit cycle.*

Proof: Let D be the compact set $\mathcal{U} \cup \gamma$. Then D is invariant since no solution in $\mathcal{U}$ can cross γ. If $\mathcal{U}$ contains no limit cycle and no equilibrium, then, for any $X \in \mathcal{U}$,

$$\omega(X) = \alpha(X) = \gamma$$

by Poincaré-Bendixson. If S is a local section at a point $Z \in \gamma$, there are sequences $t_n \to \infty$, $s_n \to -\infty$ such that $\phi_{t_n}(X), \phi_{s_n}(X) \in S$ and both $\phi_{t_n}(X)$ and $\phi_{s_n}(X)$ tend to Z as $n \to \infty$. But this leads to a contradiction of the proposition in Section 10.4 on monotone sequences. ■

Actually this last result can be considerably sharpened:

Corollary 4. *Let γ be a closed orbit that forms the boundary of an open set U. Then U contains an equilibrium point.*

Proof: Suppose U contains no equilibrium point. Consider first the case that there are only finitely many closed orbits in U. We may choose the closed orbit that bounds the region with smallest area. There are then no closed orbits or equilibrium points inside this region, and this contradicts corollary 3.

Now suppose that there are infinitely many closed orbits in U. If $X_n \to X$ in U and each X_n lies on a closed orbit, then X must lie on a closed orbit. Otherwise, the solution through X would spiral toward a limit cycle since there are no equilibria in U. By corollary 1, so would the solution through some nearby X_n, which is impossible.

Let $\nu \geq 0$ be the greatest lower bound of the areas of regions enclosed by closed orbits in U. Let $\{\gamma_n\}$ be a sequence of closed orbits enclosing regions of areas ν_n such that $\lim_{n\to\infty} \nu_n = \nu$. Let $X_n \in \gamma_n$. Since $\gamma \cup U$ is compact, we may assume that $X_n \to X \in U$. Then if U contains no equilibrium, X lies on a closed orbit β bounding a region of area ν. The usual section argument shows that as $n \to \infty$, γ_n gets arbitrarily close to β and hence the area $\nu_n - \nu$ of the region between γ_n and β goes to 0. Then the argument above shows that there can be no closed orbits or equilibrium points inside γ, and this provides a contradiction to corollary 3. ■

The following result uses the spiraling properties of limit cycles in a subtle way.

Corollary 5. *Let H be a first integral of a planar system. If H is not constant on any open set, then there are no limit cycles.*

Proof: Suppose there is a limit cycle γ; let $c \in \mathbb{R}$ be the constant value of H on γ. If $X(t)$ is a solution that spirals toward γ, then $H(X(t)) \equiv c$ by continuity of H. In corollary 1 we found an open set whose solutions spiral toward γ; thus H is constant on an open set. ■

Finally, the following result is implicit in our development of the theory of Liapunov functions in Section 9.2.

Corollary 6. *If L is a strict Liapunov function for a planar system, then there are no limit cycles.* ■

10.7 Exploration: Chemical Reactions That Oscillate

For much of the 20th century, chemists believed that all chemical reactions tended monotonically to equilibrium. This belief was shattered in the 1950s when the Russian biochemist Belousov discovered that a certain reaction involving citric acid, bromate ions, and sulfuric acid, when combined with a cerium catalyst, could oscillate for long periods of time before settling to

equilibrium. The concoction would turn yellow for a while, then fade, then turn yellow again, then fade, and on and on like this for over an hour. This reaction, now called the Belousov-Zhabotinsky reaction (the BZ reaction, for short), was a major turning point in the history of chemical reactions. Now, many systems are known to oscillate. Some have even been shown to behave chaotically.

One particularly simple chemical reaction is given by a chlorine dioxide–iodine–malonic acid interaction. The exact differential equations modeling this reaction are extremely complicated. However, there is a planar nonlinear system that closely approximates the concentrations of two of the reactants. The system is

$$x' = a - x - \frac{4xy}{1+x^2}$$

$$y' = bx\left(1 - \frac{y}{1+x^2}\right)$$

where x and y represent the concentrations of I^- and ClO_2^-, respectively, and a and b are positive parameters.

1. Begin the exploration by investigating these reaction equations numerically. What qualitatively different types of phase portraits do you find?
2. Find all equilibrium points for this system.
3. Linearize the system at your equilibria and determine the type of each equilibrium.
4. In the ab–plane, sketch the regions where you find asymptotically stable or unstable equilibria.
5. Identify the a, b-values where the system undergoes bifurcations.
6. Using the nullclines for the system together with the Poincaré-Bendixson theorem, find the a, b-values for which a stable limit cycle exists. Why do these values correspond to oscillating chemical reactions?

For more details on this reaction, see [27]. The very interesting history of the BZ-reaction is described in [47]. The original paper by Belousov is reprinted in [17].

EXERCISES

1. For each of the following systems, identify all points that lie in either an ω- or an α-limit set
   (a) $r' = r - r^2, \theta' = 1$
   (b) $r' = r^3 - 3r^2 + 2r, \theta' = 1$

(c) $r' = \sin r,\ \theta' = -1$

(d) $x' = \sin x \sin y,\ y' = -\cos x \cos y$

2. Consider the three-dimensional system

$$r' = r(1-r)$$
$$\theta' = 1$$
$$z' = -z.$$

Compute the Poincaré map along the closed orbit lying on the unit circle given by $r = 1$ and show that this closed orbit is asymptotically stable.

3. Consider the three-dimensional system

$$r' = r(1-r)$$
$$\theta' = 1$$
$$z' = z.$$

Again compute the Poincaré map for this system. What can you now say about the behavior of solutions near the closed orbit? Sketch the phase portrait for this system.

4. Consider the system

$$x' = \sin x(-0.1\cos x - \cos y)$$
$$y' = \sin y(\cos x - 0.1\cos y).$$

Show that all solutions emanating from the source at $(\pi/2, \pi/2)$ have ω-limit sets equal to the square bounded by $x = 0, \pi$ and $y = 0, \pi$.

5. The system

$$r' = ar + r^3 - r^5$$
$$\theta' = 1$$

depends on a parameter a. Determine the phase plane for representative a values and describe all bifurcations for the system.

6. Consider the system

$$x' = -y - \left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}\right)(x^3 - x)$$
$$y' = x^3 - x - \left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}\right)y.$$

(a) Find all equilibrium points.

(b) Determine the types of these equilibria.

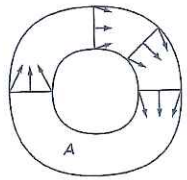

Figure 10.12 The region A is positively invariant.

(c) Prove that all nonequilibrium solutions have ω-limit sets consisting of either one or two homoclinic solutions plus a saddle point.

7. Let A be an annular region in $\mathbb{R}^2$. Let F be a planar vector field that points inward along the two boundary curves of A. Suppose also that every radial segment of A is local section. See Figure 10.12. Prove there is a periodic solution in A.

8. Let F be a planar vector field and again consider an annular region A as in the previous problem. Suppose that F has no equilibria and that F points inward along the boundary of the annulus, as before.

(a) Prove there is a closed orbit in A. (Notice that the hypothesis is weaker than in the previous problem.)

(b) If there are exactly seven closed orbits in A, show that one of them has orbits spiraling toward it from both sides.

9. Let F be a planar vector field on a neighborhood of the annular region A above. Suppose that for every boundary point X of A, $F(X)$ is a nonzero vector tangent to the boundary.

(a) Sketch the possible phase portraits in A under the further assumption that there are no equilibria and no closed orbits besides the boundary circles. Include the case where the solutions on the boundary travel in opposite directions.

(b) Suppose the boundary solutions are oppositely oriented and that the flow preserves area. Show that A contains an equilibrium.

10. Show that a closed orbit of a planar system meets a local section in at most one point.

11. Show that a closed and bounded limit set is connected (that is, not the union of two disjoint nonempty closed sets).

12. Let $X' = F(X)$ be a planar system with no equilibrium points. Suppose the flow ϕ_t generated by F preserves area (that is, if U is any open set, the area of $\phi_t(U)$ is independent of t). Show that every solution curve is a closed set.

13. Let γ be a closed orbit of a planar system. Let λ be the period of γ. Let $\{\gamma_n\}$ be a sequence of closed orbits. Suppose the period of γ_n is λ_n. If there are points $X_n \in \gamma_n$ such that $X_n \to X \in \gamma$, prove that $\lambda_n \to \lambda$. (This result can be false for higher dimensional systems. It is true, however, that if $\lambda_n \to \mu$, then μ is an integer multiple of λ.)

14. Consider a system in $\mathbb{R}^2$ having only a finite number of equilibria.

(a) Show that every limit set is either a closed orbit or the union of equilibrium points and solutions $\phi_t(X)$ such that $\lim_{t\to\infty} \phi_t(X)$ and $\lim_{t\to-\infty} \phi_t(X)$ are these equilibria.

(b) Show by example (draw a picture) that the number of distinct solution curves in $\omega(X)$ may be infinite.

15. Let X be a *recurrent* point of a planar system, that is, there is a sequence $t_n \to \pm\infty$ such that

$$\phi_{t_n}(X) \to X.$$

(a) Prove that either X is an equilibrium or X lies on a closed orbit.

(b) Show by example that there can be a recurrent point for a nonplanar system that is not an equilibrium and does not lie on a closed orbit.

16. Let $X' = F(X)$ and $X' = G(X)$ be planar systems. Suppose that

$$F(X) \cdot G(X) = 0$$

for all $X \in \mathbb{R}^2$. If F has a closed orbit, prove that G has an equilibrium point.

17. Let γ be a closed orbit for a planar system, and let $\mathcal{U}$ be the bounded, open region inside γ. Show that γ is not simultaneously the omega and alpha limit set of points of $\mathcal{U}$. Use this fact and the Poincaré-Bendixson theorem to prove that $\mathcal{U}$ contains an equilibrium that is not a saddle. (*Hint:* Consider the limit sets of points on the stable and unstable curves of saddles.)

11 Applications in Biology

In this chapter we make use of the techniques developed in the previous few chapters to examine some nonlinear systems that have been used as mathematical models for a variety of biological systems. In Section 11.1 we utilize the preceding results involving nullclines and linearization to describe several biological models involving the spread of communicable diseases. In Section 11.2 we investigate the simplest types of equations that model a predator/prey ecology. A more sophisticated approach is used in Section 11.3 to study the populations of a pair of competing species. Instead of developing explicit formulas for these differential equations, we instead make only qualitative assumptions about the form of the equations. We then derive geometric information about the behavior of solutions of such systems based on these assumptions.

11.1 Infectious Diseases

The spread of infectious diseases such as measles or malaria may be modeled as a nonlinear system of differential equations. The simplest model of this type is the SIR model. Here we divide a given population into three disjoint groups. The population of susceptible individuals is denoted by S, the infected population by I, and the recovered population by R. As usual, each of these is a function of time. We assume for simplicity that the total population is constant, so that $(S + I + R)' = 0$.