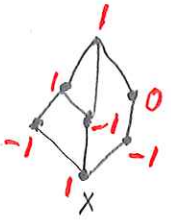
Möbius inversion formula

- The zeta function ζ of a locally finite poset P is invertible. Its inverse μ is called the Möbius function of P.
- Hence μ is determined by

$$\mu(x,x)=1,\ \forall x\in P$$

$$\mu(x,y) = -\sum_{x\le z<y}\mu(x,z)$$

Example:



$\mu(x,y)$

Prop. (Möbius inversion formula)

Let P be a poset for which every principal order ideal is finite. Let $f,g: P\to K$, where K is a field. Then

(a). $g(y)=\sum_{x\le y} f(x),\ \forall y\in P$ iff

(b). $f(y)=\sum_{x\le y} g(x)\mu(x,y),\ \forall y\in P$

Proof: $I(P,K)$ acts linearly on the vectorspace $K^P = \{f : P \to K\}$ as follows: If $\zeta \in I(P,K)$ and $f \in K^P$, then $\zeta(f) \in K^P$ is defined by

$$\zeta(f)(y) = \sum_{x \le y} f(x)\,\zeta(x,y)$$

Now if $\zeta, \eta \in I(P,K)$, then

$$(\zeta \circ \eta)(f)(y) = \zeta(\eta(f))(y) = \sum_{x \le y} \eta(f)(x)\,\zeta(x,y)$$

$$= \sum_{x \le y} \sum_{z \le x} f(z)\,\eta(z,x)\,\zeta(x,y)$$

product in $I(P,K)$

$$= \sum_{z \le y} f(z) \sum_{x \le z \le y} \eta(z,x)\,\zeta(x,y) = \sum_{z \le y} f(z)\,(\eta\zeta)(z,y)$$

$\therefore \quad \zeta \circ \eta = \eta\zeta$

In particular μ is the inverse of ζ considered as a linear tranformation $K^P \to K^P$.

The proof follows since

$$\zeta(f) = g \iff \mu(g) = f. \quad \square$$

Prop (Möbius inversion, dual form)

Let P be a poset for which $\{y : y \geq x\}$ is finite for all $x \in P$. Let $f, g : P \to K$. Then

(a). $g(x) = \sum_{y \geq x} f(y)$, $\forall\, x \in P$ iff

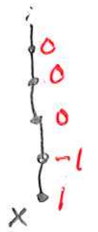
(b). $f(x) = \sum_{y \geq x} \mu(x,y) g(y)$, $\forall\, x \in P$.

Proof: Similar to the above. $\square$

- To apply Möbius inversion we want to be able to compute it for classes of posets.

Example: Suppose P is a chain, then

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y \\ -1 & \text{if } x \lessdot y \\ 0 & \text{otherwise} \end{cases}$$



Hence for $P = \mathbb{N}$, Möbius inversion says:

$$g(n) = \sum_{i=0}^{n} f(i) \quad \text{for all } n \geq 0 \iff$$

$$f(0) = g(0) \quad \text{and} \quad f(n) = g(n) - g(n-1) \quad \forall\, n > 0$$

Product theorem: Suppose P and Q are locally finite posets. If $x_1 \le_P y_1$ and $x_2 \le_Q y_2$, then

$$\mu_{P\times Q}((x_1,x_2),(y_1,y_2)) = \mu_P(x_1,y_1)\mu_Q(x_2,y_2)$$

Proof:

$$\sum_{(x_1,x_2)\le(z_1,z_2)\le(y_1,y_2)} \mu_P(x_1,z_1)\mu_Q(x_2,z_2)$$

$$= \left(\sum_{x_1 \le_P z_1 \le_P y_1} \mu_P(x_1,z_1)\right)\left(\sum_{x_2 \le_Q z_2 \le_Q y_2} \mu_Q(x_2,z_2)\right)$$

$$= \delta(x_1,y_1)\,\delta(x_2,y_2) = \delta((x_1,x_2),(y_1,y_2))$$

Hence $\mu_P(x_1,y_1)\cdot\mu_Q(x_2,y_2)$ is the Möbius function of $P\times Q$. $\square$

Example: Note that $B_n = \{0,1\}\times\dots\times\{0,1\}$

Hence $\mu(S,T) = (-1)^{|T|-|S|}$. $\square$

<u>Example</u>: Consider D_n, where $n = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k}$

if $\quad a = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \in D_n \quad$ when $a|b$, then
$\quad\;\; b = p_1^{\beta_1} \cdots p_k^{\beta_k} \in D_n$

then interval $[a,b]$ is isomorphic to $[1, \frac{b}{a}]$ which is isomorphic to a product of chains:

$$[1, \tfrac{b}{a}] \cong [0, \beta_1 - \alpha_1] \times \cdots \times [0, \beta_k - \alpha_k]$$

Hence $\mu(a,b) = \underset{\displaystyle \mu(\tfrac{b}{a})}{\overset{}{\mu(1, \tfrac{b}{a})}} = \begin{cases} (-1)^t & \text{if } \frac{b}{a} \text{ is a product of } t \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$

This is the usual Möbius-function from number theory and hence:

$$g(n) = \sum_{d|n} f(d), \quad \forall\, n|N \quad \text{iff}$$

$$f(n) = \sum_{d|n} g(d)\,\mu(n/d). \; \square$$

Suppose $\xi \in I(P,K)$ is such that $\xi^k(x,y) = 0$ for all $k \geq N$. Then

$$(\delta + \xi + \xi^2 + \dots + \xi^N)\cdot(\delta - \xi) \qquad \text{(evaluated at } (x,y))$$

$$= \delta + \xi + \xi^2 + \dots + \xi^N - \xi - \xi^2 - \dots - \xi^{N+1}$$

$$= \delta - \xi^{N+1} = \delta$$

Hence if such an N exists for all $x \leq y$, then

$$(\delta - \xi)^{-1} = \sum_{k=0}^{\infty} \xi^k$$

<u>Thm</u> (Philip Hall's theorem)
Let P be a locally finite poset. Then

$$\mu(x,y) = \sum_{k=0}^{\infty} (-1)^k c_k(x,y)$$

where $c_k(x,y)$ is the number of chains

$$x = x_0 < x_1 < x_2 < \dots < x_k = y \quad \text{in } P.$$

<u>Proof</u>: We noted before that $c_k(x,y) = (\zeta - \delta)^k(x,y)$. Set $\xi = \delta - \zeta$. Then $\xi^k(x,y) = 0$ for k sufficiently large since there is only a finite number of elements in $[x,y]$.
Hence $\mu(x,y) = \zeta^{-1}(x,y) = (\delta - \xi)^{-1}(x,y) =$

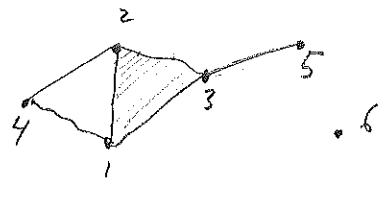
$$= \sum_{k=0}^{\infty} (-1)^k (\zeta - \delta)^k(x,y) = \sum_{k=0}^{\infty} (-1)^k c_k(x,y).$$

- An abstract simplicial complex on a finite set V is a family $\Delta \subseteq 2^V$ s.t.

  (a). $x \in V \Rightarrow \{x\} \in \Delta$

  (b). $S \subseteq T \in \Delta \Rightarrow S \in \Delta$

- The members of Δ are called faces. The dimension of a face $S \in \Delta$ is $\dim(S) = |S| - 1$



$f_{-1} = 1$
$f_0 = 6$
$f_1 = 6$
$f_2 = 1$
$f_3 = 0$

- $f_i(\Delta)$ = number of faces of dimension i

- The reduced Euler characteristic of Δ is

$$\tilde{\chi}(\Delta) = -f_{-1} + f_0 - f_1 + f_2 - \dots$$

- If P is a poset, define the order complex $\Delta(P)$ by: $V = P$ and the faces of P are the chains of P.

(suppose $x < y$ in P)

- Note that $c_k(x,y) = |\{x = x_0 < x_1 < \dots < x_k = y\}|$
$= |\{S \in \Delta(Q) : |S| = k-1\}| = f_{k-2}(\Delta(Q))$

where $Q = (x,y) = \{z \in P : x < z < y\}$ $\quad c_0(x,y) = 0$, $c_1(x,y) = 1$

Hence
$$\mu(x,y) = -1 + c_2(x,y) - c_3(x,y) + \dots$$
$$= -1 + f_0(\Delta(Q)) - f_1(\Delta(Q)) + \dots$$
$$= \tilde{\chi}(\Delta(Q)).$$

- The reduced Euler characteristic only depends on the geometric realization $|\Delta|$ of Δ.
- Hence Philip Hall's theorem enables us to use topological methods to compute or deduce properties of Möbius functions.