# Assesed homework 1.

The first set of homework is due on the Lecture where we cover "Differentiation and Distributional Derivatives" **preliminary date 18th November.** The date can change depending on how fast we progress.

**Assignemnt 1:** Show that

1. If $f(\mathbf{x}) : \mathcal{D} \mapsto \mathbb{R}$ is continuous on $\mathcal{D} \subset \mathbb{R}^n$ then $f(\mathbf{x})$ is measurable on $\mathcal{D}$.
2. If $g(\mathbf{x}) : \mathcal{D} \mapsto \mathbb{R}$ is measurable and $f(x) : \mathbb{R} \mapsto \mathbb{R}$ is continuous then $f(g(\mathbf{x}))$ is measurable. A sketch of the argument is enough.
3. If $f : [a, b] \mapsto \mathbb{R}$ is differentiable at every $x \in [a, b]$ then $f'$ is measurable.

**Assignemnt 2:** Show that there exists a minimizer $u \in L^2(\mathcal{D})$ of

$$J(u) = \int_{\mathcal{D}} \left( |u(\mathbf{x})|^2 + |u(\mathbf{x}) - f(\mathbf{x})|^{3/2} \right) d\mathbf{x}.$$

Here $\mathcal{D}$ is a bounded and measurable set and $f \in L^{3/2}(\mathcal{D})$ is a given function.

**Assignemnt 3:** Consider the following function defined on $\mathbb{R}^3$

$$f(\mathbf{x}) = \begin{cases} \frac{1}{|\mathbf{x}|} & \text{if } |\mathbf{x}| \neq 0 \\ 0 & \text{if } |\mathbf{x}| = 0. \end{cases}$$

Also let $q_j$ be an enumeration of the rationals $\mathbb{Q}^3$.

1. Show that
$$F(\mathbf{x}) = \sum_{j=1}^{\infty} \frac{f(\mathbf{x} - q_j)}{2^j} \in L^2(B_1(0)).$$
2. Show that $F(\mathbf{x})$ is not bounded on any open set $U \subset B_1(0)$.

**Assignemnt 4:** Let $\mathcal{D}$ be a domain then we say that "*$f_j$ converges in measure to $f_0$*", $f_j \hookrightarrow f_0$, if $f_j$ and $f_0$ are measurable on $\mathcal{D}$ and, for any $\epsilon > 0$,

$$\lim_{j \to 0} m\left(\{\mathbf{x} \in \mathcal{D};\ |f_j(\mathbf{x}) - f_0(\mathbf{x})| > \epsilon\}\right) = 0,$$

where $m$ is the Lebesgue measure.

Show that if

1. $\mathcal{D}$ is bounded and measurable and
2. $f_j \hookrightarrow f_0$ and
3. there exists a $g \in L^1(\mathcal{D})$ such that $|f_j(\mathbf{x})| \leq g(\mathbf{x})$

then $\|f_j - f_0\|_{L(\mathcal{D})} \to 0$ as $j \to \infty$.