Möbius functions of lattices

- Let L be a finite lattice and K a field. Consider
$$A(L,K) = \{\text{formal linear comb. of elements in } L\}$$
$$= \{\sum_{x\in L} c_x \cdot x \,;\; c_x \in K\}$$
- Make $A(L,K)$ an algebra by defining
$$x \cdot y = x \wedge y$$
- Define elements δ_x, $x \in L$, by
$$\delta_x = \sum_{y \le x} \mu(y,x) y$$
- By Möbius inversion: $x = \sum_{y \le x} \delta_y$

<u>Theorem:</u> $\delta_x \delta_y = \begin{cases} \delta_x & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$

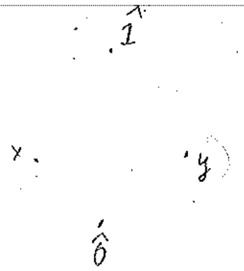
<u>Proof:</u> Consider the algebra B over K with vector-space basis $\{\xi_x \,;\, x \in L\}$ and multiplication $\xi_x \xi_y = \begin{cases} \xi_x & \text{if } x = y \\ 0 & \text{if } x \ne y \end{cases}$

Define a linear map $\theta : A(L,K) \to B$ by $\theta(\delta_x) = \xi_x$ for all $x \in L$. Hence θ is bijective and

$$\theta(x)\theta(y) = \Big(\sum_{z \le x} \xi_z\Big)\Big(\sum_{w \le y} \xi_w\Big) = \sum_{\substack{z \le x \\ w \le y}} \xi_z \xi_w = \sum_{\substack{z \le x \\ z \le y}} \xi_z =$$
$$= \sum_{z \le x \wedge y} \xi_z = \theta(x \wedge y) = \theta(xy)$$

Hence θ is an isomorphism and thus
$$\delta_x \delta_y = \theta^{-1}(\xi_x)\theta^{-1}(\xi_y) = \theta^{-1}(\xi_x \xi_y) = \begin{cases} 0 & \text{if } x \ne y \\ \delta_x & \text{if } x = y \end{cases} \quad \square$$

<u>Weisner's theorem</u>: Let L be a finite lattice and $\hat{1} \neq y \in L$. Then



$$\sum_{x : x \wedge y = \hat{0}} \mu(x, \hat{1}) = 0$$

<u>Proof</u>: We have

$$y \, \delta_{\hat{1}} = \Big(\sum_{z \le y} \delta_z \Big) \delta_{\hat{1}} = 0 \quad , \text{ since } y \neq \hat{1}$$

and thus

$$0 = y \, \delta_{\hat{1}} = y \sum_{x \in L} \mu(x, \hat{1}) x = \sum_{x \in L} \mu(x, \hat{1}) \underbrace{(y \wedge x)}_{= w}$$

$$= \sum_{w} \Big(\sum_{x : y \wedge x = w} \mu(x, \hat{1}) \Big) w$$

Look at the coefficient $w = \hat{0}$. $\square$

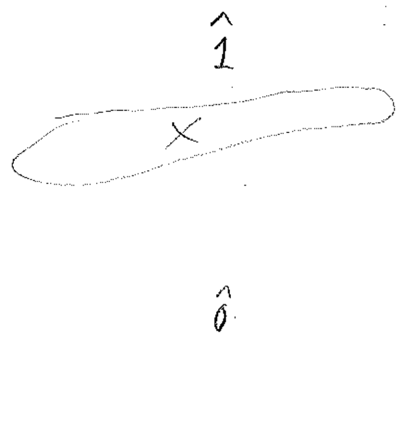
<u>Rota's Crosscut theorem</u>: Let X be a subset of L such that

(a). $\hat{1} \notin X$, and

(b). if $x \in L$ and $x \neq \hat{1}$, then $x \le y$ for some $y \in X$.

Then $\mu(\hat{0}, \hat{1}) = \sum_{k \ge 0} (-1)^k N_k$

where N_k is the number of k-subsets of X whose meet is $\hat{0}$.



<u>Proof</u>: Say $X = \{y_1, \dots, y_k\}$

$$\prod_{y \in X} (\hat{1} - y) = \prod_{y \in X} \Big(\sum_{x \in L} \delta_x - \sum_{x \le y} \delta_x \Big) = \prod_{y \in X} \Big(\sum_{x : x \not\le y} \delta_x \Big)$$

$$= \sum_{\substack{x_1 \not\le y_1 \\ \vdots \\ x_k \not\le y_k}} \delta_{x_1} \delta_{x_2} \cdots \delta_{x_k} = \sum_{\substack{x \not\le y_1 \\ \vdots \\ x \not\le y_k}} \delta_x = \delta_{\hat{1}} = \sum_x \mu(x, \hat{1}) x$$

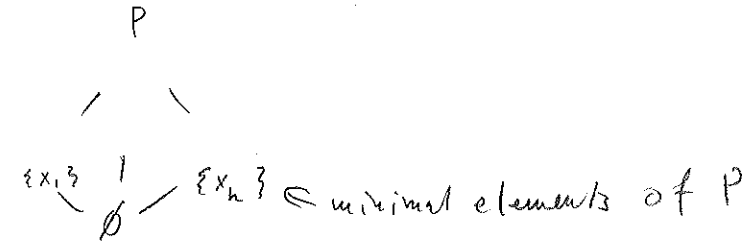
Now $\prod_{y \in X} (\hat{1} - y) = \prod_{i=1}^{k} (\hat{1} - y_i) = \sum_{S \subseteq [k]} (-1)^{|S|} \bigwedge_{i \in S} y_i$

Take the coefficient of $\hat{0}$ (which is $\mu(\hat{0}, \hat{1})$) by the above.

<u>Note</u>: X must contain all co-atoms.

<u>Corollary</u>: If $\hat{0}$ is not a meet of co-atoms, then $\mu(\hat{0}, \hat{1}) = 0$. Dually, if $\hat{1}$ is not the join of atoms, then $\mu(\hat{0}, \hat{1}) = 0$.

<u>Example</u>: $L = J(P)$.



Note that $\hat{1} = P$ is a join of atoms iff every element is minimal iff P is an antichain iff $J(P) \cong B_n$.

Hence $\mu(\hat{0}, \hat{1}) = 0$ unless P is an antichain. Moreover

$$\mu(I, J) = \begin{cases} (-1)^{|J \setminus I|}, & \text{if } J \setminus I \text{ is an antichain} \\ 0 & \text{otherwise} \end{cases}$$

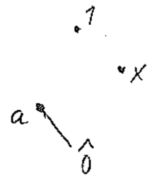
(since $[I, J] \approx J(J \setminus I)$).

Semimodular lattices

- Let a be an atom of a semimodular lattice L of rank n. Let $x \in L$ be such that $a \vee x = \hat{1}$.

(i). If $a \le x$ then $x = \hat{1}$

(ii). If $a \not\le x$, then $a \wedge x = \hat{0}$



Note:
$$\rho(x) + 1 = \rho(x) + \rho(a) \ge \rho(x \wedge a) + \rho(x \vee a) = \rho(x \wedge a) + n$$

If (ii), then $\rho(x) = n-1$, hence x is a co-atom.

(Dual Weisner's thm): $\hat{0} \ne a \in L$

$$\sum_{x : x \vee a = \hat{1}} \mu(\hat{0}, x) = 0$$

- Hence $0 = \sum_{x : x \vee a = \hat{1}} \mu(\hat{0}, x) = \mu(\hat{0}, \hat{1}) + \sum_{\substack{\text{co-atoms } b \\ \text{s.t. } b \not\ge a}} \mu(\hat{0}, b)$

$$\therefore \quad \mu(\hat{0}, \hat{1}) = - \sum_{\substack{\text{co-atoms } b \\ \text{s.t. } b \not\ge a}} \mu(\hat{0}, b) \qquad (*)$$

since $[\hat{0}, b]$ is again semimodular (of rank $n-1$)

it follows by induction that

<u>Prop.</u> The Möbius function of a semimodular lattice alternates in sign:

$$\mu(x,y)(-1)^{\rho(y)-\rho(x)} \ge 0.$$

Example: Let $V_n(q)$ be an n-dimensional vector-space over a finite field $\mathbb{F}_q$ with q elements. Let $B_n(q)$ be the modular lattice of all subspaces of $V_n(q)$ ordered by inclusion.

$B_n(q)$ is modular since

$$\dim W + \dim W' = \dim(W \cap W') + \dim(W + W')$$

Note that if $W < W'$, then

$$[W, W'] \cong B_m(q), \quad \text{where } m = \dim(W') - \dim(W)$$

$$\left(\begin{array}{l} [W, W'] \ni U \to U/W \subseteq W'/W \cong B_m(q) \\ \text{is a poset isomorphism.} \end{array} \right) \quad (**)$$

Let $a \in B_n(q)$ be an atom. $B_n(q)$ has $\binom{n}{n-1}_q = 1 + q + \dots + q^{n-1}$ coatom. By $(**)$ exactly $\binom{n-1}{n-2}_q = q^{n-2} + q^{n-3} + \dots + 1$ lie above a. Hence there are q^{n-1} coatoms b satisfying $b \not\geq a$.

Let $\mu_n = \mu_{B_n(q)}(\hat{0}, \hat{1})$. By $(*)$ and $(**)$:

$$\mu_n = -q^{n-1}\mu_{n-1}$$

Since $\mu_0 = 1$, we have $\mu_n = (-1)^n q^{\binom{n}{2}}$.

Example: What is the number of spanning subsets of $V_n(q)$

(Convention: $\emptyset$ spans no space
$\{0\}$ spans the zero-dimensional subspace $\{0\}$.) 97

For $W \in B_n(q)$, let $f(W)$ be the number of subsets of $V_n(q)$ whose span is W.

Let $g(W)$ be the number of subspaces whose span is contained in W, i.e,

$$g(W) = 2^{(q^{\dim W})} - 1$$

$$g(W) = \sum_{T \subseteq W} f(T)$$

By Möbius inversion

$$f(W) = \sum_{T \leq W} g(T)\,\mu(T, W)$$

Hence

$$f(V_n(q)) = \sum_{T \in B_n(q)} g(T)\,\mu(T, V_n)$$

$$= \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}} \left(2^{q^k} - 1\right)$$