ENUMERATIVE COMBINATORICS (SF2741), 2016
PROBLEM SET 2

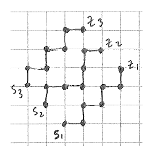
The problems are due **November 29** in class. You may discuss the problems with other students in the class, but with no one else. You may not copy solutions you have found elsewhere. If you discuss with other student(s) in the class you should mention the name(s) for each problem. Each and every one should write down your own solution in your own words. Maximal credit will be given only to complete and clear solutions.

(1) Define the *drop set* of a permutation $\pi \in \mathfrak{S}_n$ to be $\mathrm{Dr}(\pi) = \{i : \pi(i) < i\}$. Let $\gamma_n : 2^{[n]} \to \mathbb{N}$ be defined by $\gamma_n(S) = |\{\pi \in \mathfrak{S}_n : \mathrm{Dr}(\pi) = S\}|$. For $S = \{s_1 < \cdots < s_k\} \subseteq [n]$, let $\eta_n(S) = (n-k)!(s_1 - 1)\cdots(s_k - k)$. Prove that

$$\gamma_n(S) = \sum_{T \supseteq S} (-1)^{|T \setminus S|} \eta_n(T).$$

(2) Let $B(n, k, \ell)$ be the number of $k \times \ell$ matrices with entries from $\{0, 1, \ldots, n\}$ and such that each column and each row is weakly increasing. Express $B(n, k, \ell)$ as a determinant. *Hint:*

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$



(3) Problem 2.25 in the book. There are two typos in the left-hand-side of the equation displayed in 2.25b. The equation should read:

$$\sum_{m,n \geq 0} \sum_{i \geq 0} f_i(m,n) t^i \frac{x^m y^n}{m!n!} = e^{-x-y} \sum_{i \geq 0} \sum_{j \geq 0} (1+t)^{ij} \frac{x^i y^j}{i!j!}.$$

(4) A function $f : \mathbb{Z}_n \to \mathbb{Z}_k$ is *periodic* of period r, where $1 \leq r \leq n-1$, if

$$f(m+r) = f(m), \qquad \text{for all } m \in \mathbb{Z}_n.$$

Otherwise f is *non-periodic*. For $n, k \geq 1$, let $N(n,k)$ be the number of non-periodic functions $f : \mathbb{Z}_n \to \mathbb{Z}_k$ and prove

$$N(n,k) = \sum_{d|n} \mu(d,n) k^d,$$

where μ is the Möbius function of the division lattice D_n.

(5) Problem 3.129 in the book. (Hence $\sigma : P \to P$ is an order preserving bijection such that $\sigma(x) \neq x$ and $\sigma^p(x) = x$ for all $x \in P$.)
(6) Problem 3.57a in the book.
(7) Problem 3.70a and b in the book.
(8) Problem 3.90 in the book.