Hyperplane arrangements

- Let V be a finite-dimensional vectorspace over a field K, i.e., $V \cong K^n$ for some $n \in \mathbb{N}$.
- Recall that a linear hyperplane is a $(n-1)$-dimensional subspace of V, i.e., a subspace of the form

$$H = H_\alpha = \{ x \in V \,;\, \alpha \cdot x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0 \},$$

where $\alpha \in V \setminus \{0\}$.
- An affine hyperplane is a translate of a linear hyperplane, i.e.,

$$H = H_{\alpha,a} = \{ x \in V : \alpha \cdot x = a \},$$

where $\alpha \in V \setminus \{0\}$, $a \in K$.
- A hyperplane arrangement is a finite collection $\mathcal{A} = \{ H_i : i = 1, 2, \dots, m \}$ of affine hyperplanes in V.
- The intersection poset $L(\mathcal{A})$ of $\mathcal{A}$ consists of all nonempty intersections of hyperplanes in $\mathcal{A}$ (including $V = \bigcap_{i \in \emptyset} H_i$).
  The partial order is $B \leq C$ if $C \subseteq B$.
- $\mathcal{A}$ is central if $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$. If $\mathcal{A}$ is central we may translate all hyperplanes by some fixed vector so that $0 \in \bigcap_{H \in \mathcal{A}} H$. Thus we may assume

that each H is linear.

- As discussed before:

<u>Prop.</u> If A is a central hyperplane arrangement, then $L(A)$ is a geometric lattice.
Moreover for any A, each interval of $L(A)$ is a geometric lattice.

- The <u>characteristic polynomial</u> of A is

$$\chi_A(t) = \sum_{y \in L(A)} \mu(\hat{0}, y)\, t^{\dim(y)}$$

- Note that $\rho(y) = n - \dim(y)$

- We proved that for a semimodular lattice:
$\mu(x,y)(-1)^{\rho(y)-\rho(x)}$

- Hence $\chi_A(t) = t^n - a_1 t^{n-1} + a_2 t^{n-2} - \dots$

where $a_i \geq 0$

- Call any subset $B \subseteq A$ central if

$$\bigcap_{H \in B} H \neq \emptyset$$

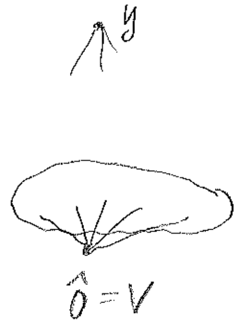
- Let $\operatorname{rank}(B)$ denote the dimension of the space spanned by the normals of the affine hyperplanes in B

- By the rank-nullity theorem
$\operatorname{rank}(B) = n - \dim(y)$, where $y = \bigcap_{H \in B} H$

Prop: Let $\mathcal{A}$ be an arrangement in $V \cong K^n$.
Then
$$\chi_{\mathcal{A}}(t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{|\mathcal{B}|}\, t^{\,n - \operatorname{rank}(\mathcal{B})}$$

Proof: We will use the dual of the Crosscut theorem.

For $y \in L(\mathcal{A})$ consider $[\hat{0}, y]$.



$X_y = \{ H \in \mathcal{A} : H \le y \ (\text{i.e., } y \subseteq H) \}$

$$\mu(\hat{0}, y) = \sum_k (-1)^k N_k(y),$$

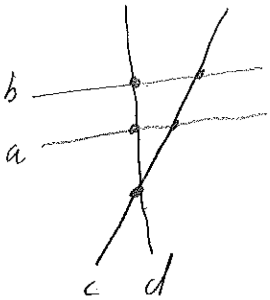
where $N_k(y)$ is the number of k-subsets of X_y whose join is y.

$$\therefore \ \mu(\hat{0}, y) = \sum_{\substack{\mathcal{B} \subseteq X_y \\ y = \bigcap_{H \in \mathcal{B}} H}} (-1)^{|\mathcal{B}|}$$

Hence
$$\chi_{\mathcal{A}}(t) = \sum_{y \in L(\mathcal{A})} \mu(\hat{0}, y)\, t^{\dim(y)} = \sum_y \sum_{\substack{\mathcal{B} \subseteq X_y \\ y = \bigcap_{H \in \mathcal{B}} H}} (-1)^{|\mathcal{B}|}\, t^{\dim(y)}$$

$$= \sum_{\mathcal{B} \text{ central}} (-1)^{|\mathcal{B}|}\, t^{\dim\left(\bigcap_{H \in \mathcal{B}} H\right)} \qquad \square$$

Example:

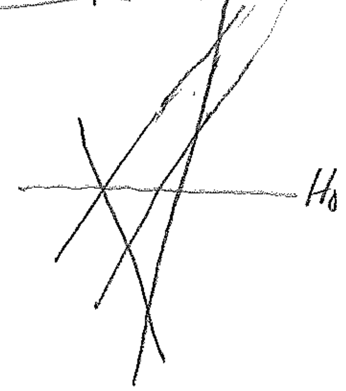
$\mathcal{A}$ 

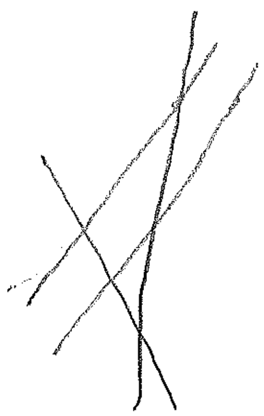
central $\mathcal{B}$	$\lvert\mathcal{B}\rvert$	$\operatorname{rank}(\mathcal{B})$	contribu.
$\emptyset$	0	0	$1 \cdot t^2$
a	1	1	$-t$
b	1	1	$-t$
c	1	1	$-t$
d	1	1	$-t$
ac	2	2	1
ad	2	2	1
bc	2	2	1
bd	2	2	1
cd	2	2	1

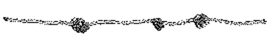
$$\chi_{\mathcal{A}}(t) = t^2 - 4t + 5$$

- Fix $H_0 \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \setminus \{H_0\}$ and $\mathcal{A}'' = \{ H \cap H_0 : H \cap H_0 \neq \emptyset,\ H \in \mathcal{A} \setminus \{H_0\} \}$. The latter is an arrangement in $H_0 \cong K^{n-1}$.

Example:

$\mathcal{A} =$ 

$\mathcal{A}' =$ 

$\mathcal{A}''$ 

Prop (Deletion-Restriction)

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t)$$

Proof: $\chi_{\mathcal{A}}(t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} (-1)^{|\mathcal{B}|} t^{\dim(\cap \mathcal{B})}$

split sum into two parts depending on if $H_0 \notin \mathcal{B}$ (Σ_1) and if $H_0 \notin \mathcal{B}$ (Σ_2).

clearly $\Sigma_1 = \chi_{\mathcal{A}'}$ by definition.

(If $\mathcal{A}$ is an arrangement where we allow hyperplanes to be the same (thus $\mathcal{A}$ is a multiset of hyperplanes), we may define

$$\chi_{\mathcal{A}}(t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{\#\mathcal{B}} x^{n-\text{rank}(\mathcal{B})}$$

Suppose $\tilde{\mathcal{A}} = \mathcal{A} \cup \{H\}$ where $H \in \mathcal{A}$.

Then $\chi_{\tilde{\mathcal{A}}} = \chi_{\mathcal{A}}$ since if $H_1 = H_2 = H$, then

$$(-1)^{|\{H_1\}|} + (-1)^{|\{H_2\}|} + (-1)^{|\{H_1, H_2\}|} = (-1)^{|\{H\}|}$$)

We could have $H_i \cap H_0 = H_j \cap H_0$ for some $i \neq j$, but by the above we may assume that this is not the case.

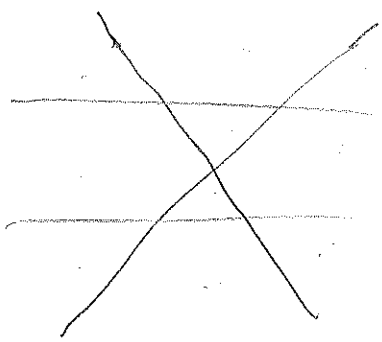
$$\Sigma_2 = \sum_{\substack{H_0 \in \mathcal{B} \subseteq \mathcal{A} \\ \text{central}}} (-1)^{|\mathcal{B}|} t^{\dim(\mathcal{B})} = \sum_{\mathcal{B}'' \subseteq \mathcal{A}''} (-1)^{|\mathcal{B}''|+1} t^{\dim(\mathcal{B}'')}$$

$$= -\chi_{\mathcal{A}''}(t) \qquad \square$$

Regions

A hyperplane arrangement in $\mathbb{R}^n$ divides $\mathbb{R}^n$ into regions i.e., the connected components of

$$\mathbb{R}^n \setminus \left(\bigcup_{H \in A} H \right)$$



$r(A) = 10$

$b(A) = 2$

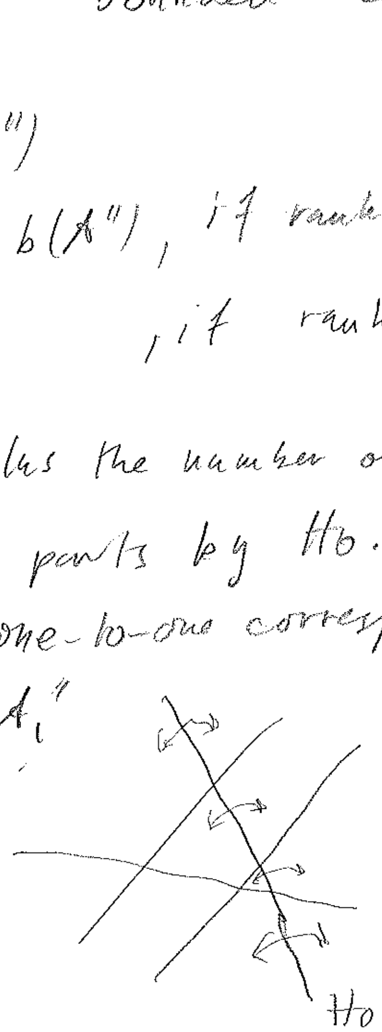
(We assume that the space spanned by the normals of the hyperplanes equals $\mathbb{R}^n$)

Let $r(A)$ be the number of regions and $b(A)$ -"- bounded regions

Lemma: $r(A) = r(A') + r(A'')$

$$b(A) = \begin{cases} b(A') + b(A''), & \text{if } \operatorname{rank}(A) = \operatorname{rank}(A') \\ 0, & \text{if } \operatorname{rank}(A) > \operatorname{rank}(A') \end{cases}$$

Proof: $r(A)$ is $r(A')$ plus the number of regions that are cut into two parts by H_0. The number of such regions are in one-to-one correspondence with the regions of A''



The second recurrence is proved similarly (Exercise).

<u>Theorem</u>: Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$.

(1). $r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1)$

(2). $b(\mathcal{A}) = (-1)^{\mathrm{rank}(\mathcal{A})} \chi_{\mathcal{A}}(1)$

<u>Proof</u>: If $\mathcal{A} = \emptyset$, then

$$r(\mathcal{A}) = 1 = (-1)^n \chi_{\mathcal{A}}(-1)$$

Now $r(\mathcal{A})$ and $(-1)^n \chi_{\mathcal{A}}(-1)$ satisfy the same recurrence, so the proof of (1) follows.

For (2), see book.