PROOFS :

Theorem 2.1

we keep the same labeling $c(e_1) \geq c(e_2) \geq \dots \geq c(e_n)$

we call $E_i = \{e_1, e_2, \dots e_i\}$ "set of elements considered up to iteration i of the algo"

we will use these two sets

$I_G \cap E_i$ intuitively, the ~~solution elem~~ solution of greedy at iteration i

$I^* \cap E_i$ and the optimal solution.

We show:

~~prove $E_i \cap I_G$ is a maximal independent subset of E_i~~

1) express $c(I_G)$ and $c(I^*)$ as functions of $|I_G \cap E_i|$ and $|I^* \cap E_i|$

2) ~~exp~~ bound $|E_i \cap I_G|$ and $|E_i \cap I^*|$ using $u(\cdot)$ and $v(\cdot)$

3) prove the theorem.

1) Note $$|E_i \cap I_G| - |E_{i-1} \cap I_G| = \begin{cases} 1 & \text{if } e_i \in I_G \text{ (it was picked by greedy)} \\ 0, & \text{otherwise} \end{cases}$$

$$c(I_G) = \sum_{e_i \in I_G} c(e_i) = c(e_1) \cdot \overbrace{|E_1 \cap I_G|}^{e_1 \overset{?}{\in} I_G} + \underbrace{\sum_{i=2}^{n} c(e_i)\left(|E_i \cap I_G| - |E_{i-1} \cap I_G|\right)}_{\text{expand}}$$

$$* = \sum_{i=1}^{n-1} |E_i \cap I_G| \cdot \left(c(e_i) - c(e_{i+1})\right) + |E_n \cap I_G| \cdot c(e_n)$$

exactly the same holds for I^*

P2

2) $E_i \cap I^*$ is independent ~~[illegible]~~ subset of E_i by definition, therefore

$$|E_i \cap I^*| \leq v(E_i)$$

We will now show that $E_i \cap I_G$ is maximal independent subset of E_i.

By contraddiction: $\exists\ e_j \in E_i \setminus I_G$ s.t. $(E_i \cap I_G) \cup \{e_j\}$ is independent. (e_j was not added by the algorithm)

consider jth iteration of step (3).

At the beginning of the iteration,

$$I \subseteq I_G$$

$$\Downarrow$$

$$I \cup \{e_j\} \subseteq \underbrace{\{E_i \cap I_G\} \cup \{e_j\}}$$

by definition, since independent, then $I \cup \{e_j\}$ independent.

$$\Downarrow$$

e_j is added by the algorithm $\square$

Hence $$|E_i \cap I_G| \geq u(E_i)$$

call $\displaystyle \rho = \max_{F \subseteq E} \frac{v(F)}{u(F)}$

$$c(I^*) \leq \sum_{i=1}^{n-1} v(E_i)\left(c(e_i) - c(e_{i+1})\right) + v(E_n)\, c(e_n)$$

$$\leq \sum_{i=1}^{n-1} \rho\, u(E_i) \cdot \left(c(e_i) - c(e_{i+1})\right) + \rho\, u(E_n) \cdot c(e_n)$$

$$\frac{v(E_i)}{u(E_i)} \leq \rho \Rightarrow v(E_i) \leq \rho\, u(E_i)$$

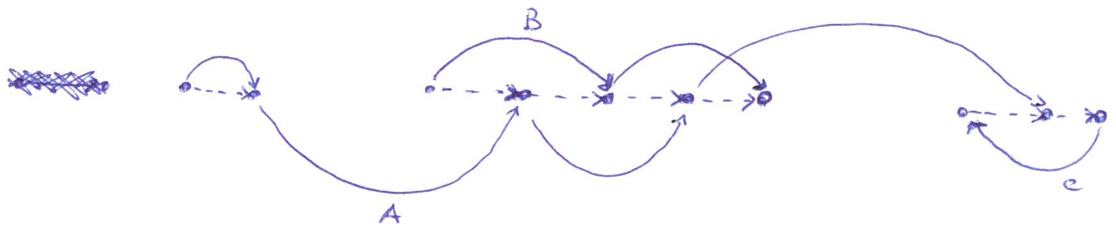
$$\leq \rho \cdot c(I_G)$$

$\square$

Proof 2.4

Let's look at one example of two maximal independent subsets of F

I ----
J ___



I is maximal, so we can't add a disjoint path

Every edge in F either

J_1 1) it shares a head with an edge in I (so we can't add it) A

J_2 2) it shares a tail with an edge in I (B)

J_3 3) it ~~shares~~ connects from the head to the tail of a maximal path in I (c)

Call J_1, J_2, J_3 the subset of edges in J with corresponding property. (the properties refer to I!)

OBS that each edge in I can share its head with at most one edge in J (otherwise J not and set disjoint paths they visit a vertex at most once!)

Same for the tail.

And each maximal path in I can be connected from head to tail by at most one edge in J.

OK!

Hence $|J_i| \le |I|$, $\forall i \in 1,2,3$.

Thus $|J| = |J_1| + |J_2| + |J_3| \le 3|I|$ $\square$

Proof 2.7

We must prove that

"if greedy algorithm produces an optimal solution", then $(E, \mathcal{I})$ is a matroid"

We prove by contradiction. Suppose $(E, \mathcal{I})$ not a matroid. then $\exists$ $F \subseteq E$ such that I and I' are two maximal independent subsets of F and $|I| > |I'|$.

We define the non-negative function c as follows

$$c(e) = \begin{cases} 1+\varepsilon, & \text{if } e \in I' \\ 1, & \text{if } e \in I \setminus I' \\ 0, & \text{otherwise} \end{cases}$$

We set $\varepsilon > 0$ and $\varepsilon < \frac{1}{|I'|}$, in this way the greedy algorithm selects the e in I', but $c(I) > c(I')$.

Hence the algorithm produces I' which is not optimal.

$\square$

Proof 2.11

for this proof we also show an example

Let $C_1, \dots C_k$ be all minimal dependent sets of $(E, \mathcal{I})$.
minimal sets among $\{F \mid F \subseteq E, F \in \mathcal{I}\}$

$\forall i \in \{1, 2, \dots k\}$ define

$$\mathcal{I}_k = \{F \subseteq E \mid C_i \not\subseteq F\}$$

Easy to verify $\mathcal{I} = \bigcap_{i=1}^{k} \mathcal{I}_i$.

Also easy to verify that ~~is a independent subset~~ $(E, \mathcal{I}_i)$ independent system.

$E = \{1, 2, 3, 4\}$

$\mathcal{I} = \{ \{1,2\}, \{2,3,4\},$
$\{1\}, \{2\}, \{3\}, \{4\},$
$\{2,3\}, \{2,4\}, \{3,4\} \}$

clearly nothing $|\cdot| = 1$

$C_1 = \{1, 3\}$
$C_2 = \{1, 4\}$

any other with cardinality ?

$\mathcal{I}_1 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,4\}, \{3,4\},$
$\{2,3\}, \{2,4\}, \{1,2,4\}, \{2,3,4\}$

$\mathcal{I}_2 = \{ \{1\} \dots \{4\}, \underbrace{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}}_{\text{always there!}}, \{1,2,3\}$
$\{2,3,4\} \}.$

We need to show that $\forall F \subseteq E$, $u(F) = v(F)$.

1) If $C_i \not\subseteq F$, then by definition $F \in \mathcal{I}_i$, and its the unique independent set.

2) If $C_i \subseteq F$, then by definition $F \notin \mathcal{I}_i$, but every $F \setminus \{u\}$, $u \in C_i$ is a maximal independent set of size $|F| - 1$

$\square$

Proof 2.12

Consider $F \subseteq E$, consider I and J two maximal independent subsets of F with respect to $(E, \mathcal{I})$.

For each $1 \le i \le k$

- let I_i be a maximal independent subset of $I \cup J$ with respect to $(E, \mathcal{I}_i)$ such that $I \subseteq I_i$

  "we take the union, so I is still ind set, so I_i exist!"

- let J_i . . . . . . such that $J \subseteq J_i$.

What do we know:

$$|J| \le |J_i| \qquad |I| \le |I_i| \qquad \forall i$$

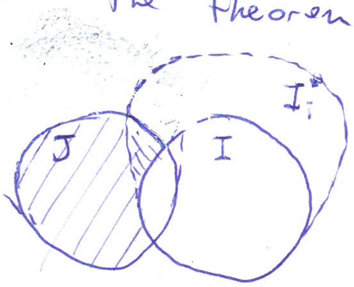
$$|I_i| = |J_i| \quad \forall i \quad , \text{ since } (E, \mathcal{I}_i) \text{ matroid}$$

we can write

$$k|J| \le \sum_{i=1}^{k} |J_i| = \overbrace{\sum_{i=1}^{k} |I_i| \le k|I| + (k-1)|J|}^{\text{needs proof } \cancel{\checkmark}}$$

$$\Longrightarrow \quad |J| \le k|I| \qquad \text{which proves the theorem.}$$

How to prove such inequality?



consider any $e \in J \setminus I$

if $e \in \bigcap_{i=1}^{k} (I_i \setminus I)$ "if e is in all I_i^*"

$$\overbrace{I \cup \{e\}}^{\text{then it is in every } \mathcal{I}_i} \in \bigcap_{i=1}^{k} \mathcal{I}_i = \mathcal{I} \quad , \text{ which contradicts the fact that } I \text{ is maximal!}$$

Hence any $e \in J \setminus I$ is in at most $k-1$ subsets $I_i \setminus I$!

$$\underbrace{\sum_{i=1}^{k} |I_i \setminus I| = \sum_{i=1}^{k} |I_i| - k|I|}_{} \le \underbrace{(k-1)|J \setminus I| \le (k-1)|J|}_{}$$

$\square$

Proof 2.22

call $\{C_1, \dots, C_m\}$ a minimum set cover $(m = opt)$

for each $j = 1, 2, \dots m$, let $\mathcal{C}_j = (C_1, \dots C_j)$

$A_i = (A_1, \dots A_i)$

call $\{A_1, \dots A_g\}$ the solution of greedy}

For each $1 \leq j \leq m$

$$f(A_{i+1}) - f(A_i) = \Delta_{A_{i+1}} f(A_i) \overset{\text{otherwise greedy. it would not be}}{\geq} \Delta_{C_j} f(A_i)$$

$$\geq \frac{1}{m} \cdot \sum_{j=1}^{m} \Delta_{C_j} f(A_i)$$

In addition

$$|S| - f(A_i) = f(A_i \cup \mathcal{C}_m) - f(A_i) = \underbrace{\sum_{j=1}^{m} \Delta_{C_j} f(A_i \cup \mathcal{C}_{j-1})}_{\text{by definition}}$$

Since f is submodular and monotone increasing we can write

$$\Delta_{C_j} f(A_i) \geq \Delta_{C_j} f(\underbrace{A_i \cup \mathcal{C}_{j-i}}_{B})$$

semi intuitive but proof in the book Proof of Lemma 2.25

Combining the three

$$f(A_{i+1}) - f(A_i) \geq \frac{1}{m}\left(|S| - f(A_i)\right)$$

multiply by -1 and add $|S|$ to both ~~mea~~ sides

$$|S| - f(A_{i+1}) \leq \left(|S| - f(A_i)\right)\left(1 - \frac{1}{m}\right)$$

Apply INDUCTION

$$|U_i| = \overbrace{|S| - f(A_i)} \leq |S| \cdot \left(1 - \frac{1}{m}\right)^i \leq |S| \cdot \left(e^{-\frac{1}{m}}\right)^i \quad *1$$

we call this U_i "elements of S not covered by A_i"

famous inequality taylor expansion (max leman)

So the size of U_i decreases from $|S|$ to 0, so there must exist an integer $i \in \{1, 2, \dots g\}$ ~~such that~~ such that $|U_{i+1}| < m \le |U_i|$, that is, after $i+1$ iterations there are at most $m-1$ elements left uncovered, so greedy will stop after at most $m-1$ more iterations. $\Rightarrow g \le i + m$ (*2)

combining *1 and *2 we obtain:

*1) $$-\frac{i}{m} \ge \ln\left(\frac{m}{|S|}\right) \Rightarrow i \le m \ln\left(\frac{|S|}{m}\right) \le m \ln \gamma$$

average size of sets in the minimum cover

*2) $$g \le i + m \le m(1 + \ln \gamma)$$

$\square$