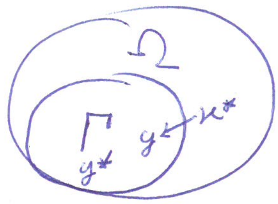
# RESTRICTION




$$\min_{x \in \Omega} f(x)$$

restrict the solutions to $x \in \Gamma \subseteq \Omega$

$\downarrow$

find optimal $y^* \in \Gamma$

OBS:

$$f(x^*) \leq f(y^*)$$

In order to analyse how good is $y^*$, we find $x^*$ (solution to original problem, therefore $x^* \in \Omega$) modify it to $y \in \Gamma$.

We compute $f(y) - f(x^*)$ to determine the performance ratio:

$$\Longrightarrow \left\| \frac{f(y^*)}{f(x^*)} \leq \frac{f(y)}{f(x^*)} = 1 + \frac{f(y) - f(x^*)}{f(x^*)} \right.$$

OBS: we find $x^*$ to evaluate the performance ratio only.

$$\max_{x \in \Omega} f(x)$$

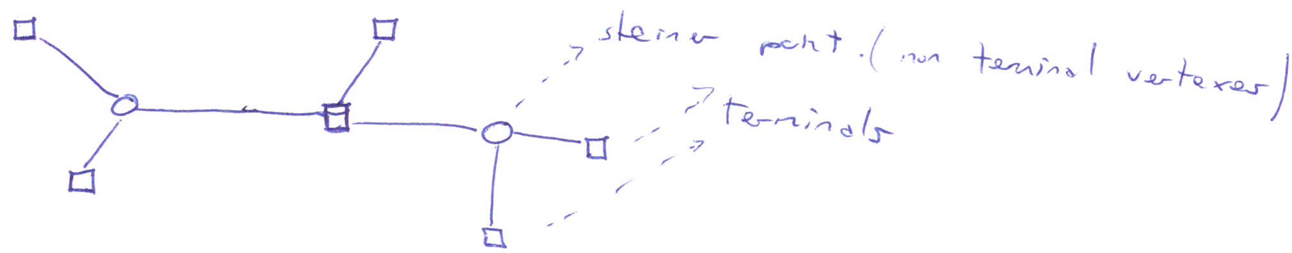
OBS: $f(x^*) \geq f(y^*)$

perf ratio $\Longrightarrow \left\| \frac{f(x^*)}{f(y^*)} \leq \frac{f(x^*)}{f(y)} = \left(1 - \frac{f(x^*) - f(y)}{f(x^*)}\right)^{-1} \right.$

# STEINER TREES



STEINER TREE: given a set of terminals, i.e, points in a metric space, any minimal (no edge can be removed) tree interconnecting all terminals is a Steiner tree.




STEINER PROBLEM: for a given set of terminals, find a shortest Steiner tree, called SMT steiner minimum tree.

<u>Network Steiner Minimum Tree (NSMT):</u> Given an edge-weigthed graph (called a network) $G = (V, E)$ and a subset $P \subseteq V$ of terminals, find a subgraph of G with the minimum total weight interconnecting all vertices in P.

Both NP-Hard. Need approximation

Obvious applications in networking.

We consider G to be complete, and the edge weight satisfy the triangle inequality.
If G not complete, we can add edges with weight that is the shortest path...

A spanning tree is a steiner tree with no Steiner points!
Can we use MST (minimum spanning tree) to approximate SMT ?

## STEINER TREES AND SPANNING TREES



~~STEINER TREE~~

~~given a set of terminals i.e. points in a metric space, any minimal (no edge can be deleted) tree interconnecting all terminals is a Steiner tree.~~

~~$\boxed{\text{SMT}}$ Steiner minimum tree, is the shortest steiner tree.~~

~~Non terminal vertexes are called steiner points.~~

~~SPANNING TREE is a steiner tree with no Steiner points.~~

Minimum Spanning tree $\boxed{\text{MST}}$ can be computed in time $O(n^2)$ (Euclidean or Rectilinear plane $\to O(n \log n)$).

Restrict the solutions to spanning trees. i.e. use a MST to approximate SMT.

P set of terminals $\frac{mst(P)}{smt(P)}$ length of MST / length of SMT

perf ratio $\| \max_{P \in \mathbb{P}} \frac{mst(P)}{smt(P)}$ $\mathbb{P}$ all input instances

**THEO 3.1**

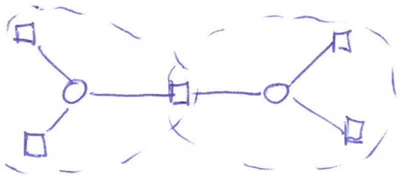
For the problem NSMT, the performance ratio of the MST approximation is equal to 2.

**Proof 3.1** — external $\square$

# K-RESTRICTED STEINER TREES



if all of its full components have size at most k



, if there is a terminal with degree higher than 1, decompose tree at that terminal → full component; every terminal is a leaf

SPANNING TREE is a 2-RESTRICTED STEINER TREE.

Consider K-RESTRICTED STEINER TREE AS approximation:
As k gets larger, the k-restricted steiner minimum tree gets closer to SMT.

what if we approximate the solution using a k-restricted steiner minimum tree (k-restricted SMT)?

**ANALYSIS:** start from $x^*$ and get $y$ (OBS $y$ is not $y^*$!)

SMT ⟶ k-restricted Steiner tree

Consider full component $T$ of size $> k$.

1. Express $T$ as a regular binary tree (exactly two children)
2. Divide binary tree in ~~an~~ subtrees of size $k$.

Doing so we can prove (SKIP THE PROOF)

**Theo 3.5** For $k \geq 2$, the k-restricted SMT is a $\left(1 + \frac{1}{\lfloor \log k \rfloor}\right)$-approximation to the SMT problem.

$$\lim_{k \to \infty} 1 + \frac{1}{\lfloor \log k \rfloor} = 1$$

, but can k-restricted SMT be computed in polynomial time?

For $k \geq 4$, computing k-rest SMT is NP-Hard!
For $k = 3$, we don't know... So, we need good approximations ALSO for the k-restricted SMT!

If interested check Sections 3.~~in~~ Chapter 3.

Now we look at another interesting problem for network design.

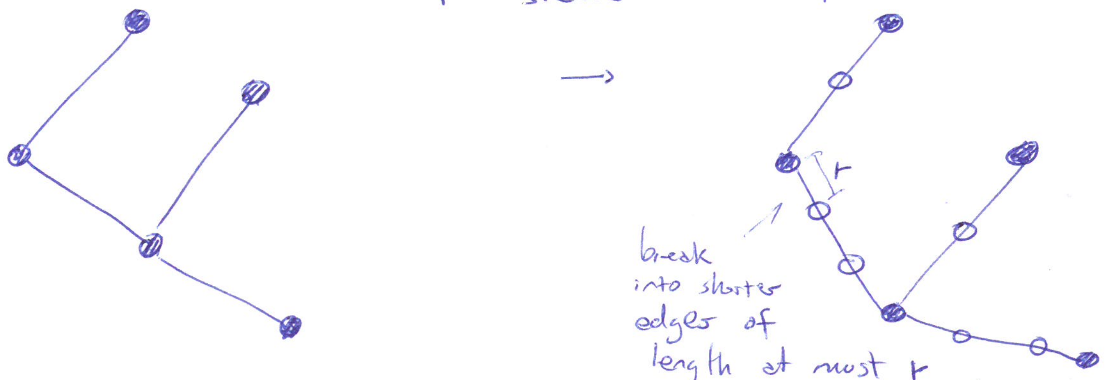
OBJ: Interconnect $n$ network nodes (in the euclidean plane) with the minimum number of antennas, subj to the constraint that each antenna can transmit up to distance $r$.

– Given $n$ terminals in the euclidean plane, and $r > 0$, find a steiner tree interconnecting all terminals with the minimum number of Steiner points, such that the length of each edge is at most $r$

ST-MSP (Steiner trees with minimum Steiner points)

So take now a spanning tree

T. $\qquad\qquad$ T′ steinerized spanning tree of T




(we add the min number of steiner points)

A minimum steinerized spanning tree is a Steinerized spanning tree with the minimum number of steiner points; can it be used as approximate solution?

<u>Lemma</u>: Let $T^*$ be a MST on a set P of terminals, and $r$ a positive real number. The steinerized spanning tree of $T^*$ is a minimum Steinerized spanning tree.

Proof: Let $T'$ arbitrary spanning tree on $P$.

Call $E(T^*)$ and $E(T')$ edge sets.

Homework: Then there is a one-to-one, onto mapping $f$ from $E(T^*)$ to $E(T')$ such that

$$\text{length}(e) \leq \text{length}(f(e)) \quad \forall\, e \in E(T^*).$$

So the Steinerised spanning tree of $T'$ has number of steiner points greater or equal to the steinerized spanning tree of $T^*$. $\square$

~~So now we know we can take a MST of P build the steinerized tree of it and use it solution~~

We will use the lemma to prove the following:

**Theo** The minimum Steinerized spanning tree is a 4-approximation for ST-MSP.

**Proof** Theorem 3.22 external $\square$