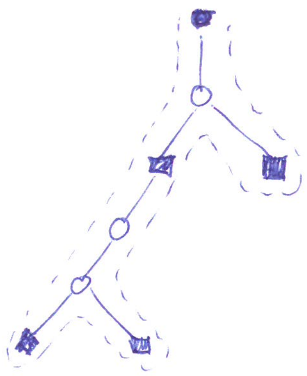
Proof 3.1

Consider SMT T (which is x^*).

There exist an Euler tour T_1 of T, which uses each edge in T twice.



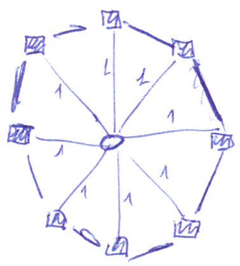
Since the graph satisfies triangle inequality the length of Euler tour is greater than length of a MST (which is y)

Hence

$$mst(P) \leq \text{length}(T_1) \leq 2 \cdot smt(P)$$

Now show that ~~ms~~ $\rho \geq 2$.

Consider graph: with $n+1$ vertexes



all other edges have weight 2

$d(0, i) = 1 \quad \forall \; i = 1, 2, \dots n$

$d(i, j) = 2 \quad \forall \; i \neq j \in \{1, 2, \dots n\}$

consider $P = \{1, 2, \dots n\}$

$smt(P) = n \qquad mst(P) = 2(n-1)$

hence $\dfrac{mst(P)}{smt(P)} = \dfrac{2(n-1)}{n} = 2 - \dfrac{2}{n}$

$\lim\limits_{n \to \infty} \dots = 2$ $\square$

Proof 3.22

We will prove the following:

"Suppose that, for any set of terminals as an input to ST-MSP, there always exists a MST with vertex degree at most d. Then the minimum Steinerized spanning tree is a (d-1)-approximation for ST-MSP."

↓

Possible to show that, for any set P of terminals in the Euclidean plane, there is a MST of P with degree at most 5. Hence the theorem follows.

Of course for specific instances of ST-MSP the ~~degg~~ maximum degree of the MST can be lower, therefore we could obtain better approx ratio.

Proof:

S^* an optimal solution (ST-MSP on input P and r) (out N^*)

k number of Steiner points in S^*.

$s_1, s_2, \ldots s_k$ steiner points in order of occurrence in a breadth-first search from a terminal point in S^*

$N(Q)$ number of steiner points in a minimum Steinerized spanning tree on Q, where Q is a set of terminals.

We will prove that we can eliminate Steiner points $s_k, s_{k-1}, \ldots s_1$, one by one, and convert S^* into a steinerized spanning tree adding at most d-1 new steiner points at each step.

Formally, for $0 \le i \le k+1$

$$N(P \cup \{s_1, \dots, s_i\}) \le N(P \cup \{s_1, \dots, s_i, s_{i+1}\}) + d - 1 \qquad *$$

we have to go from S^* to a minimum steinerized spanning tree on P. (such tree is y).

Observe that $N(P \cup \{s_1, \dots, s_k\}) = 0$, there is no need for steiner points if all the points in S^* are considered terminals.

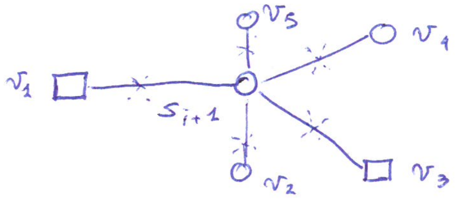
So if we prove $*$, then we get

$$N(P) \le N(P \cup \{s_1, \dots s_k\}) + k(d-1) = k(d-1)$$ which proves the theorem.

<u>Proof of</u> $*$:

consider T: a MST for $P \cup \{s_1, \dots s_i, s_{i+1}\}$, with degree $\le d$

Suppose s_{i+1} is adjacent to vertices $v_1, \dots v_j$, $j \le d$



call $d(x,y)$ euclidean distance between x and y.

Then for some $1 \le l \le j$, $d(v_l, s_{i+1}) \le r$

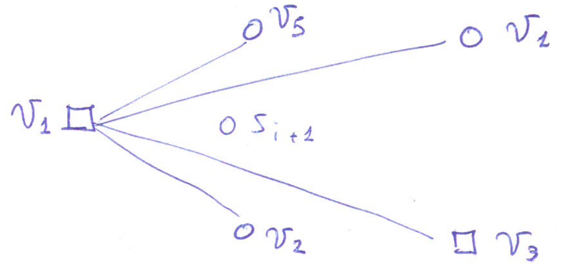
(because at least one of the vertices in $P \cup \{s_1, \dots s_i\}$ has distance at most r from s_{i+1}, we followed a breadth first ordering so we haven't created "holes").

w.l.o.g assume $d(v_1, s_{i+1}) \le r$.

We can build a spanning tree T' on $P \cup \{s_1, \dots s_i\}$ as follows:

Delete the j edges $(s_{i+1}, v_1), \dots (s_{i+1}, v_j)$

Add $j-1$ edges $(v_1, v_2) \dots (v_1, v_j)$



for each $2 \le \ell \le j$

$$\underbrace{d(v_1, v_\ell)} \underset{\text{triang ineq}}{\le} d(v_1, s_{i+1}) + d(s_{i+1}, v_\ell) \le r + \underbrace{d(s_{i+1}, v_\ell)}$$

In a steinerized spanning tree we need to break this, in edges of length $\le r$.

How many do we need? The amount we needed to break edge (s_{i+1}, v_ℓ) plus 1 degree-2 steiner point.

Hence the steinerized spanning tree induced from T' contains at most $j-1$ more Steiner points than that induced from T. Since such steinerise spanning tree from T' is also minimal, we proved ✗.

$\square$