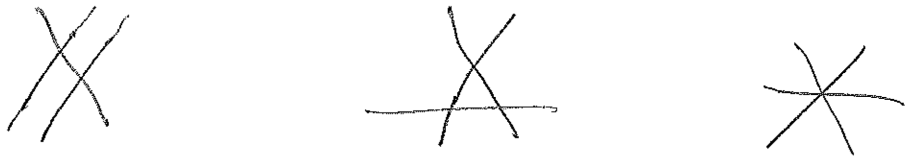
- $\{H_1, H_2, \dots, H_p\}$ are in general position if

$$\dim\left(\bigcap_{i\in S} H_i\right) = n - |S| \quad , \quad \text{for all } S \subseteq [p], \; |S| \le \min(p,n)$$



not g.p.   g.p   not g.p.

<u>Prop</u>. If $\mathcal{A}$ is an arrangement in K^n of m hyperplanes in general position, then

$$\chi_{\mathcal{A}}(t) = t^n - m t^{n-1} + \binom{m}{2} t^{n-2} - \dots + (-1)^n \binom{m}{n}$$

In particular:

$$r(\mathcal{A}) = 1 + m + \binom{m}{2} + \dots + \binom{m}{n}$$

$$b(\mathcal{A}) = (-1)^n \left(1 - m + \binom{m}{2} - \dots + (-1)^n \binom{m}{n}\right)$$

$$= \binom{m-1}{n}$$

<u>Proof</u>: Each $S \subseteq [m]$ defines a unique element $X_S = \bigcap_{i \in S} H_i \in \mathcal{L}(\mathcal{A})$. Clearly $X_S \le X_T$ in $\mathcal{L}(\mathcal{A})$ iff $X_S \supseteq X_T$ iff $S \subseteq T$. Hence $\mathcal{L}(\mathcal{A})$ is isomorph. to a truncated Boolean algebra

$$\mathcal{L}(\mathcal{A}) \cong \{S \subseteq [m] : |S| \le n\}$$

$$\chi_{\mathcal{A}}(t) = \sum_{X \in \mathcal{L}(\mathcal{A})} \mu(\hat{0}, X)\, t^{\dim(X)} = \sum_{|S| \le n} (-1)^{|S|} t^{n-|S|}$$

$\square$

Finite fields

Theorem: Let $\mathcal{A}$ be an arrangement in $\mathbb{F}_q^n$. Then

$$\chi_{\mathcal{A}}(q) = \left| \mathbb{F}_q^n \setminus \bigcup_{H \in \mathcal{A}} H \right|$$

$$= q^n - \left| \bigcup_{H \in \mathcal{A}} H \right|$$

Proof: If $X \in L(\mathcal{A})$, then $|X| = q^{\dim(X)}$.

Define $f, g : \mathcal{L}(\mathcal{A}) \to \mathbb{Z}$ by

$$f(X) = |X|$$

$$g(X) = \left| X \setminus \bigcup_{\substack{Y \in \mathcal{A} \\ Y > X}} Y \right|$$

Then $g(X)$ is the number of elements in X which are not in any other $y \in L(\mathcal{A})$ with $y \subset X$ (i.e., $Y > X$). Hence

$|X| = f(X) = \sum_{y \geq X} g(y)$. By Möbius inversion:

$$g(X) = \sum_{y \geq X} \mu(X, Y) f(y) = \sum_{y \geq X} \mu(X, y) q^{\dim(y)}$$

$$\therefore \left| \mathbb{F}_q \setminus \bigcup_{H \in \mathcal{A}} H \right| = g(\hat{0}) = \sum_{y \in \mathcal{L}(\mathcal{A})} \mu(\hat{0}, y) q^{\dim(y)} = \chi_{\mathcal{A}}(q) \qquad \square$$

Finite field method

Suppose we have an arrangement A in $\mathbb{Q}^n$.
By 'clearing of denominators we may assume that each hyperplane is of the form

$$\alpha \cdot x = a$$

where $\alpha \in \mathbb{Z}^n$ and $a \in \mathbb{Z}$.

Let $q = p^r$ be a power of a prime. By taking the α's and a's modulo p we get an arrangement A_q over $\mathbb{F}_q^n$.

Let $A = \{H_1, \ldots, H_m\}$ and $A_q = \{H_1^q, \ldots, H_m^q\}$

The characteristic polynomial only depends on for each $S \subseteq [m]$:

(a). Is $\bigcap_{i \in S} H_i \neq \emptyset$

(b). $\dim(\bigcap_{i \in S} H_i)$

- To determine (a) and (b) we perform Gaussian elimination for a system of equations

$$\alpha_1 x_1 + \cdots + \alpha_n x_n = a_1$$
$$\alpha_2' x_2 + \cdots \alpha_n' x_n = a_2$$
$$\vdots$$

- These operations are also "legal" over $\mathbb{F}_q$ unless we multiply by a number which is a multiple of p.

- Since there are only a finite number of operations in Gaussian elimination we know that if p is large enough then (a) and (b) give the same answer mod p as over $\mathbb{Q}$.

Theorem (Finite field method)

If $\mathcal{A}$ is an arrangement in $\mathbb{Q}^n$, then $\mathcal{L}(\mathcal{A}_q) \cong \mathcal{L}(\mathcal{A})$ for all $q = p^r$ with p sufficiently large. For such q

$$\chi_{\mathcal{A}}(q) = \Big| \mathbb{F}_q^n \setminus \bigcup_{H \in \mathcal{A}_q} H \Big| . \qquad (*)$$

- Since $\chi_{\mathcal{A}}$ is a polynomial $(*)$ determines $\chi_{\mathcal{A}}$.

Graphical arrangements

Let $G = ([n], E)$ be a simple graph on $[n]$, and consider the arrangement $\mathcal{A}_G = \{ H_{ij} : \{i,j\} \in E \}$, where H_{ij} is the hyperplane in $\mathbb{Q}^n$:

$$x_i - x_j = 0$$

Let $q = p$ be a large enough prime number. Then

$$\chi_{\mathcal{A}_G}(p) = \Big| \mathbb{Z}_p \setminus \bigcup_{ij \in E} H_{ij} \Big|$$

$$= |\{ f \in \mathbb{Z}_p^n : f(i) \neq f(j) \text{ for all } \{i,j\} \in E \}|$$

Hence $\chi_{\mathcal{A}_G}(p)$ counts the number of <u>proper colorings</u> of G, i.e, $\chi_{\mathcal{A}_G}$ is the chromatic polynomial of G.

• The braid arrangement is $\mathcal{B}_n = \mathcal{A}_G$, where $G = K_n$, the complete graph on $[n]$. Clearly the number of proper colorings

$$f : [n] \to [m]$$

is $m(m-1)\cdots(m-n+1)$, and hence

$$\chi_{\mathcal{B}_n}(t) = (t)_n$$

If $A \subseteq E$, then we may associate a set-partition, $S_1, S_2, \ldots, S_k$, where $S_1, \ldots, S_k$ are the connected components of the graph $([n], A)$. It is easy to see that the map

$$\mathcal{L}(\mathcal{B}_n) \ni \bigcap_{e \in A} H_e \longrightarrow S_1, \ldots, S_k \in \Pi_n$$

is an isomorphism. Hence $\mu(\hat{0}, \hat{1})$ = coefficient in front of t in $\chi_{\mathcal{B}_n}(t) = (-1)(-2)\cdots(-(n-1)) = (-1)^{n-1}(n-1)!$ as we saw before.