P-partitions

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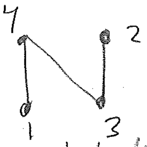
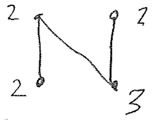
- A labelled poset is a poset P with ground set $[p]$. Write $\prec$ for partial order and $<$ for total order.



labelled poset



P-partition

$x_1^2 x_2^2 x_3^3 x_4^4$

Monomial in F_P

- A **P-partition** is a map $\sigma : P \to \mathbb{N}$ satisfying

  (i) $s \preceq t \Rightarrow \sigma(s) \geq \sigma(t)$

  (ii) $s \prec t$ and $s > t \Rightarrow \sigma(s) > \sigma(t)$

  n-P-partition

- P is naturally labeled if $s \prec t \Rightarrow s < t$.

- P is dual -"- if $s \prec t \Rightarrow s > t$

In the first case a P-partition is simply (weakly) order reversing.
In the second case a P-partitions is (strictly) order reversing.

- Fundamental generating function:

$$F_P = \sum_{\sigma \in A(P)} \prod_{t \in P} x_t^{\sigma(t)}$$

A(P) = {P-partitions}

Example:

$$\sum_{\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0} x_1^{\sigma_1} \cdots x_p^{\sigma_p}$$

$$= \sum_{\beta_1, \beta_2, \ldots, \beta_p} x_1^{\beta_1 + \ldots + \beta_p} x_2^{\beta_2 + \ldots + \beta_p} \ldots$$

$$= \frac{1}{1-x_1} \frac{1}{1-x_1 x_2} \cdots$$

$$\sum_{\sigma_p > \sigma_{p-1} > \dots > \sigma_1 \ge 0} x_1^{\sigma_1} \cdots x_p^{\sigma_p} =$$

$$= \sum x_1^{\beta_1} x_2^{\beta_1+\beta_2+1} x_3^{\beta_1+\beta_2+\beta_3+2} \cdots =$$

$$= \frac{x_2 x_3^2 \cdots x_p^{p-1}}{(1-x_p)(1-x_{p-1}x_p)\cdots(1-x_1x_2\cdots x_p)}$$

$$\sum_{\sigma_2 \ge \sigma_3 > \sigma_1 \ge \sigma_5 > \sigma_4 \ge \sigma_6 \ge 0} x_2^{\sigma_2} x_3^{\sigma_3} x_1^{\sigma_1} x_4^{\sigma_4} \cdots =$$

$$= \sum_{\beta_2, \beta_3, \dots, \beta_6} x_6^{\beta_6} x_4^{\beta_6+\beta_4} x_5^{\beta_6+\beta_4+\beta_5+1} x_1^{\beta_6+\beta_4+\beta_5+\beta_1+1} \times x_3^{\beta_6+\beta_4+\beta_5+\beta_1+\beta_3+2} x_2^{\beta_6+\beta_4+\dots+\beta_3+\beta_2+2}$$

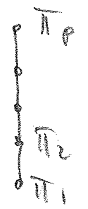
$$= x_5 x_1 x_3^2 x_2^2 \frac{1}{(1-x_2x_3x_1x_5x_4x_6)} \cdot \frac{1}{(1-x_2x_3x_1x_5x_4)} \times \cdots$$

$$\times \cdots \frac{1}{1-x_2x_3} \cdot \frac{1}{1-x_2}$$

$\pi = 231546$:

$$F_p = \prod_{i=1}^{p} x_{\pi_i}^{des(\pi_i \pi_{i+1} \cdots \pi_p)} \prod_{i=1}^{p} \frac{1}{1 - x_{\pi_1} \cdots x_{\pi_i}} \qquad (*)$$

For $\pi \in \mathfrak{S}_p$, let P_π =



$$F_{P_\pi} = (*)$$

- The <u>Jordan–Hölder set</u> $\mathcal{L}(P)$ of P is the set of linear extensions of P: $\pi \in \mathfrak{S}_p$ s.t. $\pi_i < \pi_j \Rightarrow i < j$

Lemma: $\mathcal{A}(P) = \bigsqcup_{\pi \in \mathcal{L}(P)} \mathcal{A}(P_\pi)$

Proof: Suppose that $x < y$ (as integers) but x and y are incomparable. Let P_{xy} (and P_{yx}) be the posets obtained by forcing $x < y$ ($y < x$).

$\mathcal{A}(P_{xy}) = \{\sigma \in \mathcal{A}(P) : \sigma(x) \geq \sigma(y)\}$

$\mathcal{A}(P_{yx}) = \{\sigma \in \mathcal{A}(P) : \sigma(x) < \sigma(y)\}$

$\therefore \quad \mathcal{A}(P) = \mathcal{A}(P_{xy}) \sqcup \mathcal{A}(P_{yx})$

Iterate this: $\mathcal{A}(P) = \bigsqcup_{\pi \in \mathcal{L}(P)} \mathcal{A}(P_\pi)$ $\square$

Theorem:

$$F_P = \sum_{\pi \in \mathcal{L}(P)} \quad (*)$$

Recall $D(\pi) = \{i : \pi_{i+1} > \pi_i\}$

Let $maj(\pi) = \sum_{i \in D(\pi)} i$

Let $G_P(x) = \sum_{n=0}^{\infty} a_P(n) x^n$, $a_P(n) :=$ number n-P-partitions.

Theorem: $$G_P(x) = \frac{\sum_{\pi \in \mathcal{L}(P)} x^{maj(\pi)}}{(1-x)\cdots(1-x^p)}$$

Proof: $$(*) \rightarrow \frac{x^{maj(\pi)}}{(1-x)\cdots(1-x^p)}$$

Example: P antichain

$$F_P = \sum_{\sigma_1,\dots,\sigma_p} x_1^{\sigma_1} \cdots x_p^{\sigma_p} = \frac{1}{1-x_1} \cdots \frac{1}{1-x_p}$$

$$\Rightarrow \quad G_P(x) = \frac{1}{(1-x)^p}$$

$$G_P(x) = \frac{\sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{maj}(\pi)}}{(1-x)\cdots(1-x^p)}$$

$$\therefore \quad \sum x^{\mathrm{maj}(\pi)} = \frac{1-x}{1-x} \cdot \frac{1-x^2}{1-x} \cdots \frac{1-x^p}{1-x}$$

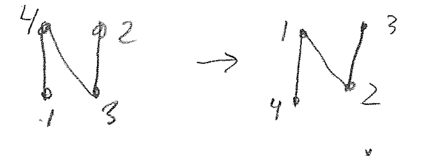
$$= (1+x)(1+x+x^2)\cdots(1+x+\dots+x^{p-1})$$

$$= \underline{(n)}_x$$

Hence maj has the same distribution as inv.

Reciprocity

Let P^* be the poset on $[p]$ defined by
$s < t$ in P iff $p+1-s < p+1-t$ in P^*



P $\qquad$ P^*

Theorem: $x_1 \cdots x_p F_{P^*}(x_1, \dots, x_p) = (-1)^p F_P(x_p^{-1}, \dots, x_1^{-1})$

Proof: If $\pi = \pi_1 \cdots \pi_p$, let $\pi^* = (p+1-\pi_1)\cdots(p+1-\pi_p)$

Then $\pi \in \mathcal{L}(P) \iff \pi^* \in \mathcal{L}(P^*)$

$$\prod x_{\pi_i^*}^{des(\pi_i^* \cdots \pi_p^*)} \prod_{i=1}^{p} \frac{1}{1 - x_{\pi_1^*} \cdots x_{\pi_i^*}}$$

$$= \prod x_{\pi_i^*}^{p-i-des(\pi_i \cdots \pi_p)} \prod -\!\!\parallel\!\!-$$

$$= x_1^{-1} \cdots x_p^{-1} \underbrace{x_{\pi_1^*}^{p} \cdots x_{\pi_p^*}^{1}}_{= \prod_{i=1}^{p} x_{\pi_1^*} \cdots x_{\pi_i^*}} \prod x_{\pi_i^*}^{-des(\pi_i \cdots \pi_p)} \prod -\!\!\parallel\!\!-$$

$$\frac{x_{\pi_1^*} \cdots x_{\pi_i^*}}{1 - x_{\pi_1^*} \cdots x_{\pi_i^*}} = -\frac{1}{1 - x_{\pi_1^*}^{-1} \cdots x_{\pi_i^*}^{-1}}$$

$\blacksquare$

The order polynomial of P, $\Omega_P(m)$,

$\Omega_P(m) := \#$ P-partitions $\sigma : P \to [m]$

Let $e_k(P) := \#$ surjective P-partitions $\sigma : P \to [k]$

$\therefore$ $\Omega_P(m) = \sum_{k=1}^{p} e_k(P) \binom{m}{k}$

Ex:

P_π $\quad 3\ 5 \overset{>}{\ } 2 \overset{>}{\ } 1\ 6 \overset{>}{\ } 4$

$$m \geq \sigma(3) \geq \sigma(5) > \sigma(2) > \sigma(1) \geq \sigma(6) > \sigma(4) \geq 1$$

$$\begin{array}{cccccc} -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & & \\ -1 & -1 & -1 & & & \end{array}$$

multi-choose 6 out of $m-3$

$$\left(\!\!\binom{m-3}{6}\!\!\right)$$

In general $\Omega_{P_\pi}(m) = \left(\!\!\binom{m - des(\pi)}{p}\!\!\right)$

Hence $\Omega_P(m) = \sum_{\pi \in \mathcal{L}(P)} \left(\!\!\binom{m - \mathrm{des}(\pi)}{p}\!\!\right)$

Note that $\sum_m \left(\!\!\binom{m-d}{p}\!\!\right) x^m = \ldots$

$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k} = (-1)^k \binom{-n}{k}$

$$\left(\begin{aligned} \left(\!\!\binom{m-d}{p}\!\!\right) &= \binom{m-d+p-1}{p} = \binom{m-d+p-1}{m-d-1} \\ &= \left(\!\!\binom{p+1}{m-d-1}\!\!\right) = \\ &= (-1)^{m-d-1} \binom{-p-1}{m-d-1} \end{aligned} \right)$$

$$= \sum_m (-1)^{m-d-1} \binom{-p-1}{m-d-1} x^{m-d-1} x^{d+1}$$

$$= x^{d+1} \sum_k \binom{-p-1}{k} (-x)^k = \frac{x^{d+1}}{(1-x)^{p+1}}$$

Let $H_P(x) = \sum_{m \geq 0} \Omega_P(m) x^m$

$A_P(x) = \sum_{\pi \in \mathcal{L}(P)} x^{\mathrm{des}(\pi)+1}$

<u>Theorem</u>: $H_P(x) = \dfrac{A_P(x)}{(1-x)^{d+1}}$

<u>Ex</u>: P antichain $\Omega_P(m) = m^p$ $\sum_m m^p x^m = \dfrac{A_p(x)}{(1-x)^{p+1}}$

$$x^p G_{p*}(x) = (-1)^p G_p(1/x)$$

$$W_{p*}(x) = x^{\binom{p}{2}} W_p(1/x)$$

$$H_{p*}(x) = (-1)^{p+1} H(1/x)$$

$$A_{p*}(x) = x^{p+1} A_p(1/x)$$

Corollary: $\Omega_{P^*}(m) = (-1)^p \Omega_P(-m)$

Proof:

$$\left(\!\!\binom{m - des(\pi^*)}{p}\!\!\right) = \left(\!\!\binom{m - (p-1-des(\pi))}{p}\!\!\right)$$

$$= \binom{m + des(\pi)}{p} = (-1)^p \left(\!\!\binom{-m - des(\pi)}{p}\!\!\right) \quad \blacksquare$$

Let P be naturally labelled and graded of rank r.

Define $\theta : \mathcal{A}(P) \to \mathcal{A}(P^*)$ by

$$\theta(\sigma)(s) = \sigma(s) + r - \rho(s)$$

θ bijection and $\theta^{-1}(\sigma^*)(s) = \sigma^*(s) - r + \rho(s)$;

$\sigma(t) \geq \sigma(s)$, t over s $\xrightarrow{\theta}$ t over s, $\sigma(t) + r - \rho(s) - 1 > \sigma(s) + r - \rho(s)$

$\sigma(t) > \sigma(s)$, t over s $\xrightarrow{\theta^{-1}}$ t over s, $\sigma(s) - r + \rho(s) + 1 \geq \sigma(s) - r + \rho(s)$

$\theta(\sigma)(\text{maximal}) = \sigma(\text{maximal})$
$\theta(\sigma)(\text{minimal}) = \sigma(\text{minimal}) + r$

$$A_m(P) := \{\sigma \in \mathcal{A}(P) : \sigma : P \to [m]\}$$

$$\theta : A_m(P) \leftrightarrow A_{m+r}(P^*)$$

$$\therefore \quad \Omega_P(m) = \Omega_{P^*}(m+r) = (-1)^p \Omega_P(-m-r)$$

$$\boxed{\Omega_P(-m) = (-1)^p \Omega_P(m-r)}$$