Practice Exam SF1677 Fall 2016

Total marks 35: The relationship between the marks and grades are

$$\mathbf{A}: 31 \qquad \mathbf{B}: 27 \qquad \mathbf{C}: 23 \qquad \mathbf{D}: 19 \qquad \mathbf{E}: 15 \qquad \mathbf{FX}: 14.$$

A G^+ on the first homework assignment corresponds to full mark (5 marks) on question 1 and a G^+ on the second homework assignment corresponds to full mark (5 marks) on question 2.

All your answers should be proved unless otherwise stated.

Question 1: Let $h : [0, 1) \mapsto \mathbb{R}$ be uniformly continuous. Prove that there exists a unique continuous $g : [0, 1] \mapsto \mathbb{R}$ such that $g(x) = h(x)$ for every $x \in [0, 1)$.

(5 marks)

Question 2: Let $f_j \rightrightarrows f_0$ in the set $S \subset \mathbb{R}^n$ and assume that $x_j \to x_0 \in S$ for some sequence $x_j \in S$ and that f_0 is continuous at x_0. Prove that $f_j(x_j) \to f_0(x_0)$.

(5 marks)

Question 3: Assume that $f_j \rightrightarrows f_0$ on $[a, b]$ and that $f_0 \in C^0([a, b])$. Prove that for every $\epsilon > 0$ there exists a J_ϵ and a partition $P = \{a = x_0 < x_1 < ... < x_N = b\}$ such that

$$j > J_\epsilon \Rightarrow U(f_j, P) - L(f_j, P) < \epsilon.$$

We do not assume that any of the functions f_j is continuous or integrable.

(5 marks)

Question 4:

1. State and prove the intermediate value theorem. Make sure that you mention when you use any property of the real numbers.
2. Provide a counterexample to the intermediate value property for continuous functions $f : \mathbb{Q} \mapsto \mathbb{R}$.

(5 marks)

Question 5: Define the concepts: *metric space* and *compactness*. Then prove, or disprove, that

$$M = \{f \in C^0([a, b]);\ |f(x) - f(y)| \leq \sqrt{|x - y|} \text{ for every } x, y \in [a, b],\ f(a) = 0\}$$

is a compact space under the metric $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$. You may use any theorem from the course, without proof, as long as you state it correctly and show that the assumptions are verified.

(5 marks)

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Question 6: Let $S \subset [0,1]$ be the set of real numbers such that the decimal expansion (in base 10) does not contain any 1 or 9. For instance $0.3333... \in S$ but $0.34923... \notin S$. Prove that S is a zero set.

(5 marks)

Question 7: Let $\mathcal{A} \subset C^0([0,1])$ be a set such that

1. For every $x, y \in [0,1]$ and every $a, b \in \mathbb{R}$ there exists a function $f \in \mathcal{A}$ such that $f(x) = a$ and $f(y) = b$.
2. For any finite set $f_1, f_2, ..., f_n \in \mathcal{A}$ the function $g(x) = \max(f_1(x), f_2(x), ..., f_n(x)) \in \mathcal{A}$.

Let $h \in C^0([0,1])$ and $\epsilon > 0$. Prove that there exists a function $f_\epsilon \in \mathcal{A}$ such that $f_\epsilon(1/2) = h(1/2)$ and $f_\epsilon(x) \geq h(x) - \epsilon$ for every $x \in [0,1]$.

(5 marks)

Good Luck!