Catalan numbers

- The Catalan numbers are
$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$$

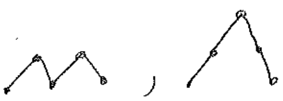
$1, 1, 2, 5, 14, 42, 132$

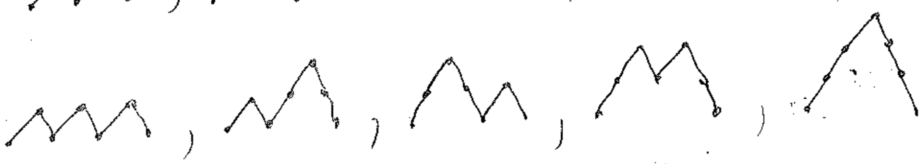
- C_n is frequently occurring in combinatorics

- A Dyck-path of semi-length n is a path from $(0,0)$ to $(2n,0)$ using steps $u=(1,1)$ and $d=(1,-1)$ which never goes below the x-axis.

$n=0$. (the empty Dyck-path)

$n=1$ 

$n=2$ 

$n=3$ 

- Note that a non-empty Dyck path of semilength $n+1$, may be uniquely written as
$P = u P_1 d P_2$, where P_1 and P_2 are Dyck-paths

first time P reaches the x-axis.

- Let $|P|$ = semilength of P

Hence if $|P_1| = k$, then $|P_2| = n-k$

$$\therefore \quad D_{n+1} = \sum_{k=0}^{n} D_k D_{n-k}, \qquad n = 0,1,2,\dots, \quad (*)$$

where D_n is the number of Dyck-paths of semi-length n.

Let $D(x) = \sum_{n=0}^{\infty} D_n x^n$

Then by $(*)$;

$$D(x) = 1 + x \sum_{n=0}^{\infty} D_{n+1} x^n = 1 + x \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} D_k D_{n-k} \right) x^n$$

$$= 1 + x\, D(x)^2$$

Solving this gives

$$D(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

which sign is right?
It is only $-$ that gives a power series!

$$\therefore \quad D(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

Now $\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \sum_{m=0}^{\infty} \binom{1/2}{m} (-4)^m x^m$

$$\binom{\frac{1}{2}}{m} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right) \cdots \left(\frac{1}{2}-m+1\right)}{m!} = \left(\frac{1}{2}\right)^m \frac{1 \cdot (1-2)(1-2\cdot 2) \cdots (1-2(m-1))}{m!}$$

$$= \left(\frac{1}{2}\right)^m (-1)^{m-1} \frac{1 \cdot 3 \cdot 5 \cdots (2(m-1)-1)}{m!}$$

Note that

$$(2(m-1))! = 1\cdot 3\cdot 5 \cdots (2(m-1)-1)\cdot(2\cdot 1)(2\cdot 2)\cdots(2(m-1))$$
$$= 1\cdot 3\cdot 5\cdots(2(m-1)-1)\cdot 2^{m-1}(m-1)!$$

Hence
$$\binom{1/2}{m} = \left(\tfrac{1}{2}\right)^m (-1)^{m-1}\left(\tfrac{1}{2}\right)^{m-1}\frac{(2(m-1))!}{(m-1)!\,m!}$$
$$= -2\left(\tfrac{-1}{4}\right)^m C_{m-1}$$

and $\binom{1/2}{0} = 1$

$$\therefore\ D(x) = \frac{1-\sqrt{1-4x}}{2x}$$
$$= \frac{-\sum_{m\geq 1}\binom{1/2}{m}(-4)^m x^m}{2x} = \frac{\sum_{m\geq 1} 2C_{m-1}x^m}{2x} = \sum_{m=0}^{\infty} C_m x^m$$

- We have proved $D_n = C_n$.

- A permutation $\pi \in \mathfrak{S}_n$ is 132-avoiding if there is no $1 \leq i < j < k \leq n$ s.t. $\pi(i) < \pi(k) < \pi(j)$.
- Denote the set of 132-avoiding permutations in $\mathfrak{S}_n$ by $\mathfrak{S}_n(132)$.

- If $\pi \in \mathfrak{S}_{n+1}$, write π as $\pi = L\,(n+1)\,R$

the subword to the left of $n+1$.

the subword to the right of $n+1$,

- Note that $\pi = L(n+1)R \in \mathfrak{S}_{n+1}$ is 132-avoiding iff

  (1). Each letter of L is greater than all letters of R, and

  (2). L and R are 132-avoiding

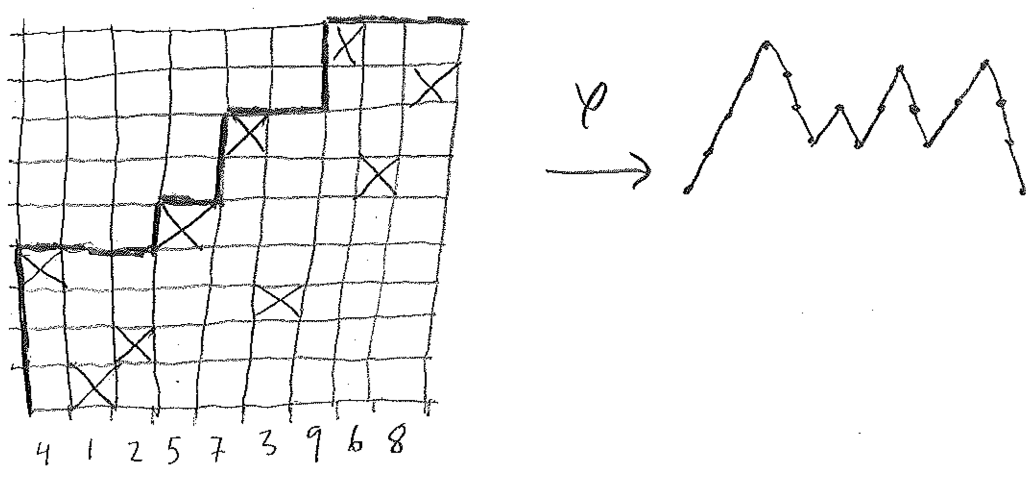
- Hence $$|\mathfrak{S}_{n+1}(132)| = \sum_{k=0}^{n} |\mathfrak{S}_k(132)| \cdot |\mathfrak{S}_{n-k}(132)|,$$

  $|\mathfrak{S}_0(132)| = 1$.

- Hence $|\mathfrak{S}_n(132)| = C_n$.

- A permutation $\pi \in \mathfrak{S}_n$ is 321-avoiding if there is no $1 \le i < j < k \le n$ s.t. $\pi(i) > \pi(j) > \pi(k)$.

- We may represent a permutation by a diagram



- φ defines a bijection since the entries below the border must be in increasing order.

$\therefore$ $|\mathfrak{S}_n(321)| = C_n$.