ENUMERATIVE COMBINATORICS (SF2741), 2016
PROBLEM SET 3

The problems are due **January 13** (by email or in my mailbox at KTH). You may discuss the problems with other students in the class, but with no one else. You may not copy solutions you have found elsewhere. If you discuss with other student(s) in the class you should mention the name(s) for each problem. Each and every one should write down your own solution in your own words. Maximal credit will be given only to complete and clear solutions. Problems 1–5 give 4 points each, and 6–7 give 6 points each.

(1) Problem 3.62 c), d), e) of the book. *Hint:* See p. 27–28 of the book.
(2) Problem 3.89 of the book.
(3) Problem 3.114 b) of the book.
(4) Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be an integer partition. Consider the arrangement $\mathcal{A}_\lambda$ in $\mathbb{R}^n$ defined by

$$\begin{aligned} x_i &= x_j, \text{ for all } 1 \leq i < j \leq n, \\ x_1 &= j_1, \text{ for all integers } 0 \leq j_1 \leq \lambda_1, \\ x_2 &= j_2, \text{ for all integers } 0 \leq j_2 \leq \lambda_2, \\ &\vdots \\ x_n &= j_n, \text{ for all integers } 0 \leq j_2 \leq \lambda_n. \end{aligned}$$

Find a simple formula for the characteristic polynomial $\chi_{\mathcal{A}_\lambda}(t)$.
Note that there are $\binom{n}{2} + \lambda_1 + \lambda_2 + \cdots + \lambda_n + n$ hyperplanes in $\mathcal{A}_\lambda$.

(5) Let m and n be positive integers. Consider the labeled poset $P_{m,n}$ on $[m+n]$ with relations $m \prec m+1 \prec \cdots \prec m+n$ and $m \prec m-1 \prec \cdots \prec 2 \prec 1$. Determine the order polynomial $\Omega_{P_{m,n}}(x)$.
(6) Consider the Ferrers board $B_n = \{(i,j) : 1 \leq i \leq n, 1 \leq j \leq 2i\}$ (see Ch. 2.4 of the book).
   (a). Determine the number of ways of placing out n non-attacking rooks on B_n.
   (b). Determine the number of ways of placing out n non-attacking rooks on B_n so that the rooks are in increasing order, i.e., if (i_1, j_1) and (i_2, j_2), where $i_1 < i_2$, are occupied by rooks, then $j_1 < j_2$.
   (c). (harder) Determine the number, $A_{n,k}$, of ways of placing out n non-attacking rooks on B_n so that exactly k of the rooks are at positions with odd y-coordinate (j-coordinate).
(7) Let P be a finite naturally labelled poset.
   (a). In this problem we consider a labeled poset to be a poset with ground set a subset of the set of positive integers (not just $[p]$ as in the notes). Prove that if x and y are independent variables, then

$$\Omega_P(x+y) = \sum_{I \in J(P)} \Omega_I(x)\Omega_{P \setminus I}(y),$$

   (We define $\Omega_\emptyset(x) := 1$). Here I and $P \setminus I$ are the induced labeled sub-posets with ground sets I and $P \setminus I$. *Hint:* It suffices to prove it for positive integers x, y, since both sides of the equation are polynomials.
   (b). For $x \in \mathbb{C}$, let ξ_x the element of the incidence algebra $I(J(P), \mathbb{C})$ defined by $\xi_x(I,J) = \Omega_{J \setminus I}(x)$, and note that $\xi_0 = \delta$ and $\xi_1 = \zeta$. Prove $\xi_x \xi_y = \xi_{x+y}$ for all $x, y \in \mathbb{C}$.
   (c). Deduce that the Möbius function of $J(P)$ is

$$\mu(I,J) = \xi_{-1}(I,J) = \Omega_{J \setminus I}(-1) = \begin{cases} (-1)^{|J \setminus I|} & \text{if } J \setminus I \text{ is an antichain,} \\ 0 & \text{otherwise.} \end{cases}$$

   Hint: Use reciprocity to evaluate $\Omega_{J \setminus I}(-1)$.

(d). Prove the recursion

$$\Omega_P(x) = \sum_{S \subseteq M(P)} (-1)^{|S|} \Omega_{P \setminus S}(x+1),$$

where $M(P)$ is the set of minimal elements of P.