Department of Mathematics

SF1626 Calculus in Several Variable
Exam
Tuesday, January 10, 2017

Skrivtid: 08:00-13:00
Allowed aids: none
Examinator: Mats Boij

The exam consists of nine problems each worth a maximum of four points.

Part A of the exam consists of the first three problems. To the score from part A your bonus points are added. The total score from part A can never exeeed 12 points. The bonus points are added automatically and the number of bonus points can be seen from the result page.

The three following problems make up part B and the three last problems part C, which is primarily intended for the higher grades.

The thresholds for the grades will be given by

Grade	A	B	C	D	E	Fx
Total score	27	24	21	18	16	15
Score on part C	6	3	–	–	–	–

In order to acheive full credit on a problem the solution has to be well presented and easy to follow. This means in particular that all notation should be defined, the logical structure clearly described in words or symbols and that the arguments are well motivated and clearly explained. Solutions that suffer seriously regarding this can not acheive more than two points.

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DEL A

1. A particle moves in such a way that its position is described by
$$(x, y, z) = (t\cos t, t\sin t, t),$$
where the axes measure units in meter and where t is the time measured in seconds, starting from 0.
   (a) Determine the velocity of the particle after 1 second. **(1 p)**
   (b) What is the speed of the particle have after 1 second? **(1 p)**
   (c) What is the arc length of the curve that the particle describes during the first second? **(2 p)**

2. Determine whether or not the vector field $\mathbf{F}(x, y) = (\sin x \sin y, -\cos x \cos y)$ is conservative and then compute the integral
$$\int_C \sin x \sin y\, dx - \cos x \cos y\, dy$$
along the curve C from $(0, 0)$ to $(1, 1)$ that is given by $y = x^2$. **(4 p)**

3. Consider the function $f(x, y) = 2x^3 - 6xy^2 + y^4$.
   (a) Show that $(0, 0)$, $(3, -3)$ and $(3, 3)$ are the only critical points of f. **(1 p)**
   (b) Find the Taylor polynomial of degree two of f at the critical point $(3, -3)$. Use them to decide whether the points are local minima, local maxima or saddle points. **(2 p)**
   (c) Explain why the Taylor polynomial of degree two at the origin cannot be used to determine whether the origin is a local minimum or maximum or a saddle point. **(1 p)**

DEL B

4. Let D be the domain in the plane that is described by the inequalities
$$x^2 + y^2 \leq 1 \quad \text{and} \quad x \leq |y|.$$
Determine the centre of gravity of the domain D. **(4 p)**

5. *The hyperbolic coordinates* of the plane are defined by
$$(u, v) = \phi(x, y) = \left(\ln \sqrt{\frac{x}{y}}, \sqrt{xy}\right) \quad \text{for } x,\ y > 0.$$
   (a) Compute the Jacobian matrix of ϕ. **(2 p)**
   (b) Use a linear approximation in order to approximate the value of $\phi(1{,}01,\ 0{,}99)$. **(2 p)**

6. It has been observed that the fish in a certain fishing area is always swimming according to a certain flow that can be described by the vector field
$$\mathbf{F}(x, y, z) = (1, -2xy, 2xz) \quad \text{[kilogramme fish per minute and unit area]},$$
where x and y are the geographic coordinates and $z \leq 0$ the depth.
   (a) Give the equations that determine the flow lines of the vector field. **(1 p)**
   (b) A plane quadratic net can be either fixed between the points
$$A: \quad (0, 0, 0),\ (0, 1, 0),\ (0, 1, -1),\ (0, 0, -1),$$
   or
$$B: \quad (0, 1, 0),\ (1, 1, 0),\ (1, 1, -1),\ (0, 1, -1).$$
   Which of the alternatives must be chosen in order to maximise the catch, and how much fish will have been caught with that particular net if it has been left out for the duration of 5 hours? **(3 p)**

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DEL C

7. Let f be the function defined by
$$f(x,y,z) = x + y + z$$
on the domain
$$\mathcal{D}(f) = \left\{(x,y,z) \in \mathbb{R}^3 : xy = 1 \text{ och } x^2 + y^2 + z^2 = 4\right\}.$$
Determine the largest and the smallest value that f assumes on the domain $\mathcal{D}(f)$.
(4 p)

8. Green's theorem is an important tool for computing line integrals.
   (a) State Green's theorem. Do not forget to include all conditions. **(1 p)**
   (b) Prove Green's theorem in the special case where the curve is given as the positively oriented boundary of the unit square, i.e. of the domain defined by $0 \leq x \leq 1$ and $0 \leq y \leq 1$. **(3 p)**

9. If two points (x_1, y_1) and (x_2, y_2) are chosen uniformly at random within the unit circle we can compute the average area of the triangle given by the two points together with the origin as
$$\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^1\int_0^1 \frac{r_1 r_2}{2}\left|\sin(\theta_1 - \theta_2)\right| \, r_1 r_2 dr_1 dr_2 d\theta_1 d\theta_2.$$
Compute this average area. **(4 p)**