KTH Teknikvetenskap

SF1624 Algebra och geometri
Exam
Wednesday, 11 January 2017

Time: 08:00-11:00
No books/notes/calculators etc. allowed
Examiner: Tilman Bauer

This exam consists of six problems, each worth 6 points.

Part A comprises the first two problems. The bonus points from the seminars will be automatically added to the total score of this part, which however cannot exceed 12 points.

The next two problems constitute part B, and the last two problems part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

To get full score on a problem, your solution must be well-presented and easy to follow. In particular, you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument. Solutions severely lacking in these respects will achieve at most 3 points.

PART A

1. (a) For which values of k does the system of equations (in x, y, and z)

$$\begin{aligned} kx + ky + z &= 3 \\ 2x + ky + z &= 2 \\ 4x + 3y + 3z &= 8 \end{aligned}$$

have a unique solution, no solution, or infinitely many solutions? **(4 p)**

(b) Solve the system for $k = 1$. **(2 p)**

2. Let P be the plane in $\mathbb{R}^3$ which contains the point $(1, 0, 0)$ and the line $t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, t in $\mathbb{R}$.

(a) Find a normal vector and an equation for P. **(3 p)**

(b) Find the orthogonal projection of the point $(2, 4, 2)$ to the line which goes through the origin and the point $(0, 1, -1)$. **(3 p)**

Part B

3. Let $T = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ be the base change matrix from the basis $\mathcal{V}$ to the basis $\mathcal{W}$ of a subspace U of $\mathbb{R}^4$.

(a) Find a base change matrix from the basis $\mathcal{W}$ to the basis $\mathcal{V}$. **(3 p)**

(b) Let $f\colon U \to U$ be a linear map such that $[f]_{\mathcal{W}} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$. Find $[f]_{\mathcal{V}}$. **(3 p)**

(Here $[f]_{\mathcal{B}}$ denotes the matrix for the map f with respect to the basis $\mathcal{B}$.)

4. A linear map $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ is defined by the formula $L(\vec{x}) = \vec{e}_3 \times \vec{x}$ for all vectors $\vec{x}$. Here $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\times$ denotes the cross product.

(a) Find the standard matrix for $L\colon \mathbb{R}^3 \to \mathbb{R}^3$. **(2 p)**

(b) L transforms the plane $x_3 = 0$ onto itself. Describe geometrically how vectors in this plane are transformed. **(1 p)**

(c) Determine all eigenvalues and corresponding eigenvectors for L. **(3 p)**

Part C

5. Let Q be the quadratic form on $\mathbb{R}^{2n}$ defined by

$$Q(x_1, \ldots, x_{2n}) = x_1x_{2n} + x_2x_{2n-1} + \cdots + x_nx_{n+1}.$$

(a) Find the symmetric matrix associated with Q. **(2 p)**

(b) Determine the type of Q: positive/negative (semi)definite or indefinite? **(4 p)**

6. Let A be an $n \times n$-matrix. $\mathrm{Col}(A)$ denotes the column space of A. Show:

(a) If $\mathrm{Col}(A^k) = \mathrm{Col}(A^{k+1})$ for some integer $k \geq 1$ then $\mathrm{Col}(A^k) = \mathrm{Col}(A^{k+l})$ all integers $l \geq 1$. **(3 p)**

(b) If $A^j = 0$ for some integer $j \geq 1$ then $A^n = 0$. **(3 p)**