SF2722 Differential Geometry, spring 2017
Homework 1

Examiner: Mattias Dahl, Hans Ringström.
Code of conduct (Hederskodex): It is assumed that:

- you shall solve the problems on your own and write down your own solution,
- if you in spite of this are using something you have gotten from somewhere else for some reason (a friend, a book or the internet etc.) you should give a reference to the source.

Deadline: Hand in your solutions before Tuesday January 31 (we will not accept solutions handed in after Tuesday February 7).

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1. Let N denote the *north pole* $(0,\dots,0,1) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, and let S denote the *south pole* $(0,\dots,0,-1)$. Define the *stereographic coordinates* $\sigma : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1,\dots,x^{n+1}) = \frac{(x^1,\dots,x^n)}{1-x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

a) For any $x \in \mathbb{S}^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u,0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace.

b) Show that σ bijective, and

$$\sigma^{-1}(u^1,\dots,u^n) = \frac{(2u^1,\dots,2u^n,|u|^2-1)}{|u|^2+1}.$$

c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \setminus \{N\}, \sigma)$ and $(\mathbb{S}^n \setminus \{S\}, \tilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$.

d) Show that this smooth structure is the same as the one defined in Example 1.31, page 20, in Lee's book.

(Lee Problem 1-7 on page 30.)

2. Let $F : \mathbb{S}^3 \to \mathbb{S}^2$ be defined by

$$F(w,z) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w}),$$

where we think of $\mathbb{S}^3$ as the subset $\{(w,z) \mid |w|^2 + |z|^2 = 1\}$ of $\mathbb{C}^2$. Compute sufficiently many coordinate representations of F to prove that it is smooth.

(Lee Problem 2-3c on page 48.)

3. Let M be the set of circles of positive radius, as well as degenerate circles of radius zero (points), contained in $\mathbb{S}^2 \subset \mathbb{R}^3$. Find an atlas for M, giving it the structure of a smooth manifold with boundary. Try to identify what M is diffeomorphic to.