EP2200
Queueing theory and teletraffic systems

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Recitation 1 – Probability theory overview

Probability Theory Essentials

Outline

- Events
- Probability
- Random variables and distributions
- Moments
- Linear combination, min, max
- Important distributions

Z, Laplace transform: later

Sample space, Event

S : sample space, set of possible outcomes of an experiment

- discrete
- continuous

e : outcome of an experiment $e \in S$

E : event, a subset of S

Combination of events

Union, "or" : $E \cup F = \{e \in S \mid e \in E \vee e \in F\}$

Intersection, "and" : $E \cap F = \{e \in S \mid e \in E \wedge e \in F\}$

Complement, "not" : $\bar{E} = \{e \in S \mid e \notin E\}$

Difference : $E - F = \{e \in S \mid e \in E \wedge e \notin F\}$

Probability

N : number of experiments

N(E) : number of outcomes in E

$$P(E) = \lim_{N\to\infty} \frac{N(E)}{N}$$

Axioms:

$0 \leq P(E) \leq 1$

$P(S) = 1$

if $\underbrace{E \cap F}_{\text{mutually exclusive}} = \emptyset$, then $P(E \cup F) = P(E) + P(F)$

Properties:

$P(\emptyset) = 0$

$P(\bar{E}) + P(E) = 1$

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$

if $E \subseteq F$, then $P(E) \leq P(F)$

Conditional probability

Probability of event E, given that F has accured.

Def: $P(E|F) = \dfrac{P(E \cap F)}{P(F)}$ $\Longrightarrow$ $P(E \cap F) = P(E|F)P(F)$

conditional probability

joint probability $(P(E,F))$

Independence

Events E and F are independent if and only if

Def: $P(E \cap F) = P(E) \cdot P(F)$

$\Longrightarrow P(E|F) = \dfrac{P(E \cap F)}{P(F)} = P(E)$

Law of total probability

Complete set of mutually exclusive events $\{E_1, E_2 \dots E_n\}$

$$\bigcup_i E_i = S, \quad E_i \cap E_j = \emptyset \quad \forall E_i, E_j$$

then

for any event F: $F = F \cap S = F \cap (\bigcup_i E_i) = \bigcup_i (F \cap E_i) \Rightarrow$

$$P(F) = P(\bigcup_i (F \cap E_i)) = \sum_{i=1}^{n} P(F \cap E_i) = \sum_{i=1}^{n} P(F \mid E_i) P(E_i)$$

Bayes' formula

To calculate $P(E_i \mid F)$ when we know $P(F \mid E_j)$-s:

$$P(E_i \mid F) = \frac{P(E_i \cap F)}{P(F)} = \frac{P(F \mid E_i) P(E_i)}{\sum_{j=1}^{n} P(F \mid E_j) P(E_j)}$$

Random variables and distributions

Events do not always have an easy to capture notation

$\Rightarrow$

Random variable X is a mapping: $\underbrace{X : S}_{\text{state space}} \rightarrow \underbrace{\mathbb{R}}_{\text{real numbers}}$

Often the mapping is trivial: eg., temperature, number of students...

$X \left\langle \begin{array}{l} \underline{\text{discrete}} \left\langle \begin{array}{l} \text{finite} \\ \text{infinite} \end{array} \right. \quad (\text{discrete state space}) \\ \underline{\text{continuous}} \quad (\text{continuous state space}) \end{array} \right.$

Characterization of distributions

Cumulative distribution function, (CDF): $F_X(x) = P(X \leq x)$

Complementary CDF (CCDF): $\bar{F}_X(x) = P(X > x) = 1 - F_X(x)$

Probability density function (PDF) of continuous r.v.:

$$f_X(x) = \frac{dF_X(x)}{dx} = \lim_{dx \to 0} \frac{F_X(x+dx) - F_X(x)}{dx} = \lim_{dx \to 0} \frac{P(x < X \leq x+dx)}{dx}$$

Probability mass function (PMF) of discrete r.v:

$$p_X(x) = P(X = x), \quad \sum_i p_X(x_i) = 1$$

Moments and central moments

1st moment: Expectation: $\bar{x} = E[X] = \int_{-\infty}^{\infty} x f(x) dx$

$\sum_i x_i p_i$

nth moment: $E[X^n] = \int_{-\infty}^{\infty} x^n f(x) dx$

$\sum_i x_i^n p_i$

Variance (second central moment) $\sigma^2 = Var[X] = E[(X - \bar{X})^2] = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x) dx = E[X^2] - E[X]^2$

↳ standard deviation $\sigma = \sqrt{Var[X]}$

Skewness (third central moment) ...

Some useful properties

$E[cX] = c\,E[X]$ why? $\quad E[cX] = \int_{-\infty}^{\infty} c x f(x)\,dx = c \int_{-\infty}^{\infty} x f(x)\,dx \ldots$

$E[X_1 + X_2 \cdots] = E[X_1] + E[X_2] \ldots \rightarrow$

¡Note! $E[(X_1+X_2)^2] \neq E[X_1^2] + E[X_2^2]$!

$Var[cX] = c^2\,Var[X]$

$Var[X_1 + X_2 \cdots] = Var[X_1] + Var[X_2]$ only if X_1 and $X_2 \ldots$ are independent.

Combination of random variables

$f(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) \qquad \alpha_1 + \alpha_2 = 1$

$E[X] = \alpha_1 E_1[X] + \alpha_2 E_2[X]$

$E[X^2] = \alpha_1 E_1[X^2] + \alpha_2 E_2[X^2]$

Famous distributions

Discrete: $\boxed{\text{Poisson distribution}}$

PMF:

$$P(X=k) = p_k = \frac{\lambda^k}{k!} e^{-\lambda} \qquad k = 0, \dots, \infty$$

CDF: not nice

Mean: $$\underline{\underline{E[X]}} = \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \cdot \lambda \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^{k-1}}{k!}}_{e^{\lambda}} = \underline{\underline{\lambda}} \qquad \sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$$

Variance $\mathrm{Var}[X] = \lambda$

Continuous : $\boxed{\text{Exponential}}$

PDF: $f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

CDF: $F(x) = 1 - e^{-\lambda x}$

Mean: $E[x] = \int_0^\infty x \cdot e^{-\lambda x} \, dx = \cdots = \frac{1}{\lambda}$

$Var[x] = \frac{1}{\lambda^2}$

$\boxed{\text{Erlang-k}}$

Sum of k Exp. distributed rv: $X = X_1 + X_2 + X_k, \quad X_i = Exp(\lambda)$

PDF: $f(x) = \frac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x}$

$E[x] =$ sum of id. r.v. $= \frac{k}{\lambda}$

$Var[x] =$ — " — $= \frac{k}{\lambda^2}$