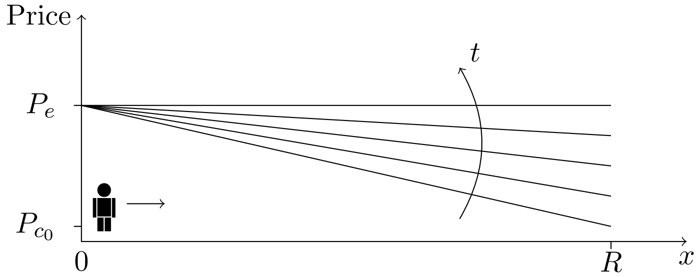
Homework 1

Analytical solutions of the Navier–Stokes equations

due January 24, 23:59

Task 1

A man goes to a large, round market to buy fish. He does not have much money so he wants to spend as little as possible. His problem is that the fish stands at the edge of the market have higher prices than those at the centre: the price decreases linearly from the edge to the centre. To complicate his life, the price at the centre of the market rises with time, and so does the price in the rest of the market, always changing linearly from edge to centre.



Given that:

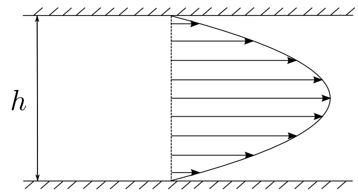
- the market is a circle with radius $R = 5$ km;
- the man can walk at a maximum speed of 2.5 km/h;
- the price at the edge of the market is fixed to $P_e = 300$ kr/kg;
- at the start, $t = 0$, the price of fish at the centre of the market is $P_{c_0} = 10$ kr/kg;
- in one hour the price at the centre has risen such that the price is the same in the whole market: $P_f = 300$ kr/kg.

At what time and distance from the edge of the market will the man be able to buy the cheapest fish? How many kilograms of fish will he be able to buy if he has 100 kr with him?

(*Hint:* use the definition of the material derivative: the price changes because the man is moving in a non-uniform 'price field' and because time passes)

[Solution: he will buy approximately 0.379 kg of fish]

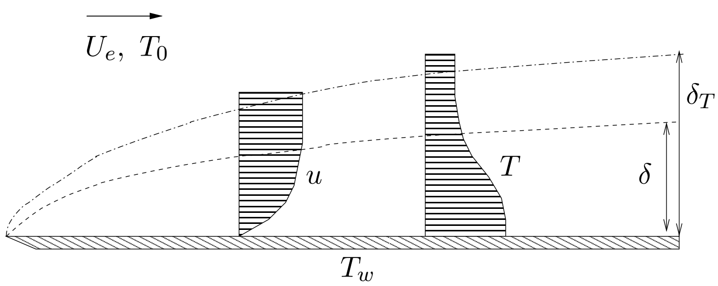
Task 2



Consider the fluid flow between two fixed infinite parallel plates, where a pressure gradient $\frac{\partial p}{\partial x} = C$ is imposed in the direction parallel to the plates. Find a steady solution to the incompressible Navier–Stokes equations for this flow case, *i.e.* so-called plane Poiseuille flow.

(Hint: Assume that the flow is fully developed, meaning $\frac{\partial}{\partial x} = 0$ for all velocity components and that body forces are absent)

Task 3



Consider the incompressible flow with free-stream velocity U_e over a semi-infinite flat plate. Let the incoming flow have a temperature T_0 while the plate is kept at the greater temperature T_w. The temperature field around the plate obeys the following equation:

$$\rho c_p \frac{DT}{Dt} = \kappa \nabla^2 T$$

where κ is the thermal conductivity, ρ the density and c_p the specific heat at constant pressure. Assume that the Péclet number, $Pe = RePr$, is large so that a thin thermal boundary layer develops on the plate.

Derive the appropriate boundary-layer approximation of the temperature equation in the limit of large Péclet number and estimate the thermal boundary-layer thickness δ_T.

(Hint: Assume a steady flow and that the ratio between the boundary layer thicknesses of flow velocity and temperature scales as $\delta/\delta_T \sim \mathcal{O}(1)$)