Recitation 4

Chapter 3:

$\boxed{3.6}$

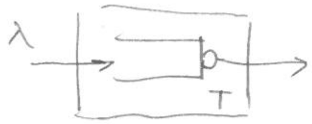
Chapter 4:

$\boxed{4.1}$

$\boxed{4.5}$

Make-up exam problem: 2013-06-13, problem 1.

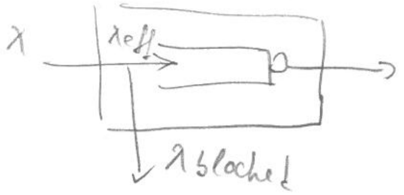
- Littles theorem:
- relation between the arrival rate of customers, average number of customers in the system and the mean time that the customers spend in the system



N - avg. num. of cust in the system
Nq - " - " - " - " - in the queue
Ns - " - " - " - " - under service
$\overline{T}$ - avg. time spent in the system
W - " - " - " - in the queue
$\overline{x}$ - avg. service time

$\overline{N} = \lambda \cdot \overline{T}$
$N_q = \lambda \cdot W$
$N_s = \lambda \cdot \overline{x}$
$T = \overline{x} + W$

- Useful because
  - nothing is assumed about the system
  - the arrival process can be anything
- Littles theorem for system with blocking



$N = \lambda_{eff} \cdot T$
T - avg system time for accepted customers

- Kendal notation: $A/B/m/c/p/\sigma$

A - arrival process
B - serving process
m - number of servers
c - system capacity (buffers + servers)
p - population
σ - service order

- Offered load $a = \lambda \cdot x = \frac{\lambda}{\mu}$ [Erlang]
- Server utilization in systems with infinite buffer capacity, m servers

$\rho = \frac{a}{m}$ (stability requires $\rho < 1$)

- Steady state probs. has to satisfy:

$\underline{p} \cdot \underline{Q} = \underline{0}$

Q - transition rate matrix

3.6 - serve 1, store 2 packets (altogether) → including the one served (1.)

- Poisson arrival
- Service: 2 tasks, each $X_i \sim Exp(\mu)$, $\frac{1}{\mu} = \frac{1}{30}$ [usec]

$P_0 = 0.6$

T - average time spent in the node = ?

⇒ We need arrival rate λ to answer the question and the state probabilities ⇒ MC

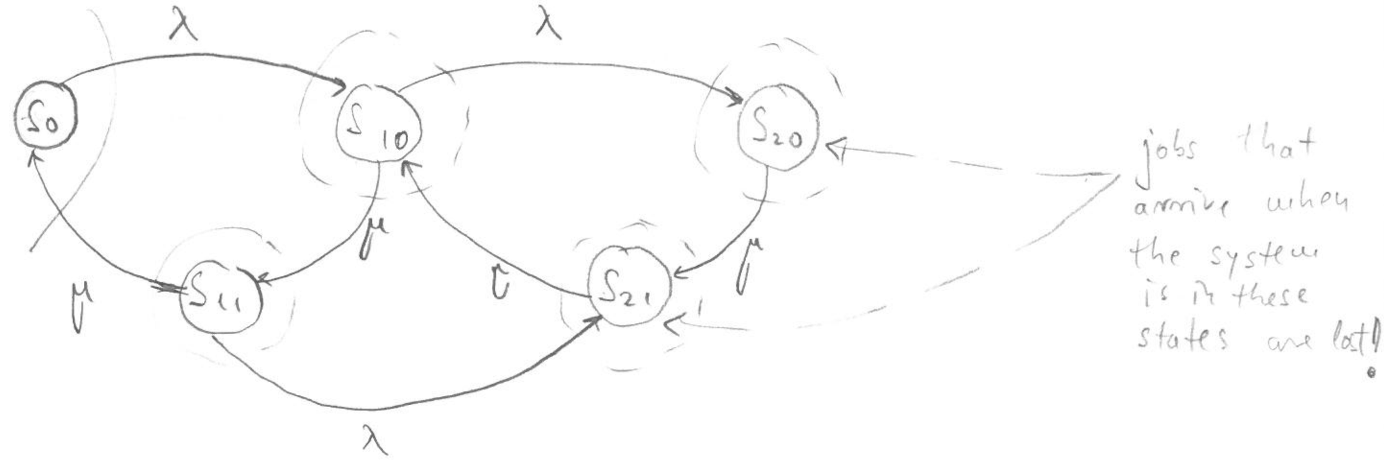
S_0 : empty

S_{10} : 1 packet, error check

S_{11} : 1 packet, transmission

S_{20} : 2 packets, one in the buffer, one error check

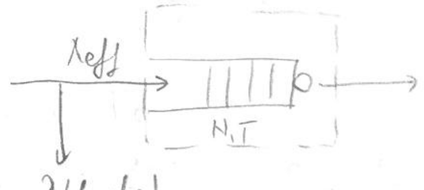
S_{21} : 2 packets, one in the buffer, one transmission



- Balance eq:

$P_0 = 0.6$

$$\left.\begin{aligned}
&P_0 \lambda = P_{11} \mu \\
&P_{10} \cdot (\lambda + \mu) = P_0 \lambda + P_{21} \mu \\
&P_{11} \cdot (\lambda + \mu) = P_{10} \cdot \mu \\
&P_{20} \cdot \mu = P_{10} \cdot \lambda \\
&P_{21} \cdot \mu = P_{11} \cdot \lambda + P_{20} \cdot \mu \\
&P_0 + P_{10} + P_{11} + P_{20} + P_{21} = 1
\end{aligned}\right\}
\quad P_0 = f\left(\frac{\lambda}{\mu}\right) \Rightarrow \lambda \approx 7.63$$



2.

$$T = \frac{\bar{N}}{\lambda_{eff}} = \frac{0 \cdot p_0 + 1 \cdot (p_{10} + p_{11}) + 2 \cdot (p_{20} + p_{21})}{[1 - \underbrace{(p_{20} + p_{21})}_{p_{blocked}}] \cdot \lambda} = 75.4 \text{ ms}$$

Alternative solution:

$S_0 \rightarrow T_0 = \{\text{own service}\} = 2 \cdot \frac{1}{\mu}$

$S_{11} \rightarrow T_{11} = \{\text{transmission + own service}\} = 3 \cdot \frac{1}{\mu}$

1. under transmission and new arrival comes

$S_{10} \rightarrow T_{10} = \{\underbrace{\text{error check + transmission}}_{\text{previous service}} + \text{own service}\} = 4 \cdot \frac{1}{\mu}$

$$a_0 = \frac{p_0}{1 - (p_{20} + p_{21})}$$

(It is not possible to enter the system if it is in S_{20} or S_{21})

$$a_{11} = \frac{p_{11}}{1 - (p_{20} + p_{21})}$$

$$a_{10} = \frac{p_{10}}{1 - (p_{20} + p_{21})}$$

$$T = a_0 \cdot T_0 + a_{11} \cdot T_{11} + a_{10} \cdot T_{10}$$

4.1 Poisson process, single output link (1.)

arrival process, serving process, population, service order

Kendall notation: A / B / m / c / p / σ

m: number of servers; c: system capacity (buffers + servers)

a.) - packet lenghts exponentially distributed ⇒ service time exponentially distributed

- buffer capacity ∞

- M / M / 1

b.) - packet lenght is fixed ⇒ service time is fixed (deterministic distr.)

- buffer can store n packets

M / D / 1 / n+1

c.) the packet length is L with prob. p_L, and l with prob. p_l.

- no buffer at the node

M / G / 1 / 1

4.5 population - 300 employees

$\lambda_i = \frac{2}{60} \frac{calls}{min}$

$\frac{1}{\mu_i} = 4.5 \, min$ $\quad n = 90 \leftarrow$ number of servers (outgoing links)

a.) offered load = arrival rate × service time

$$a = \sum_{i=1}^{300} \lambda_i \cdot \frac{1}{\mu_i} = 300 \cdot \frac{2}{60} \cdot 4.5 = 45 \, [Erlang]$$

b.) utilization of the outgoing links:

$$\underbrace{\rho = \frac{a}{n}}_{\downarrow} = \frac{45}{90} = 0.5$$

$\lambda \cdot \frac{1}{\mu \cdot m} \Rightarrow$ (in the case of the infinite buffer capacity)

EP2200 Queuing Theory and Teletraffic Systems

Monday, June 3rd, 2013.

Available teacher: Vitkoria Fodor

Allowed help: Calculator, Beta mathematical handbook or similar, provided sheets of queuing theory formulas, Laplace transforms and Erlang tables.

1. You decide to model a stochastic system with a three state continuous time Markov chain, with states S_1, S_2 and S_3. The system can move from S_1 to S_2 or to S_3, from S_2 to S_3 and from S_3 to S_1.

   a) Does this system has stationary solution? Motivate your answer. (1p)

   b) The transition rates are $q_{12} = 1$, $q_{13} = 1$ $q_{23} = 1$ $q_{31} = 1$. Give the Q matrix and the matrix equation of the stationary solution. Calculate the stationary state distribution. (3p)

   c) Assume, the system is in S_1 at time zero. Give the distribution of time the system stays in S_1, before moving to another state. Prove or motivate your answer. Give the probability that the system does not move to another state in the first 2 time units. (3p)

   d) The system is actually an ice cream machine. In S_1 it is in perfect state and produces 100 ice creams per time unit. In S_2 it is half clogged, but still produces 40 ice creams per time unit. Finally, in S_3 it needs to be stopped and cleaned. Considering the parameters in part b, what is the average time of cleaning? How many ice creams are produced per time unit in average? (3p)

2. You would like to move to the cloud computing business with low prices (and on small scale). For that you need to dimension your servers very well, such that you can provide fast response with the delay users still accept. For that you model the cloud computing service with a queuing model. You consider exponential service time, with an average of 12sec, and you assume that in average one request will arrive per second, according to a Poisson process. You decide to install $N = 15$ servers, so your cloud service can serve N requests at the same time. If all servers are busy, requests wait in a buffer that is not limited. Waiting requests are served in FIFO order.

   a) Define the queuing model of the system, and draw its Markov chain. (2p)

   b) What is the offered load in your system? What is the utilization of a server? Is the system stable? Motivate your answer. (2p)

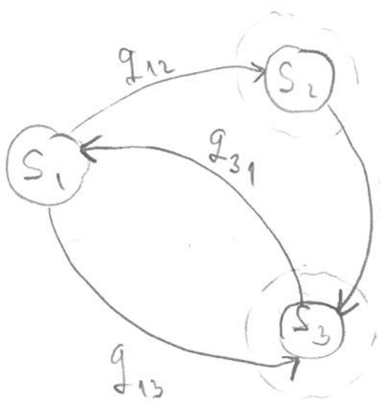
   c) Give the probability, that an arriving request can not be served immediately and the average time the users need to wait for answer (that is, waiting plus service time). (2p)

   d) Consider a request that arrives when all the servers are busy and there are 2 requests waiting. Give the distribution of the waiting time of this arriving request in transform form and in time domain. (4p).

June 3rd, 2013 Problem 1

a.) Conditions for stationary solution:

- irreducible (there is a path between any two states) ✓
- finite number of states => for all states the mean time to return to the state is finite ✓



d.) Average time of cleaning:

$Exp(q_{31}) \Rightarrow \bar{\tau}_3 = \frac{1}{q_{31}} = 1$ time unit

Average production:

$\bar{N} = 100 \cdot p_1 + 40 p_2 + 0 \cdot p_3 = 35$

b.)

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

From matrix equation:

$\underline{p} \cdot \underline{\underline{Q}} = \underline{0}$

$$[p_1 \ p_2 \ p_3] \cdot \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{array}{ll} -2 \cdot p_1 + 0 \cdot p_2 + 1 \cdot p_3 = 0 \Rightarrow & p_3 = 2p_1 \\ 1 \cdot p_1 - 1 \cdot p_2 + 0 \cdot p_3 = 0 & p_2 = p_1 \\ p_1 \cdot 1 + p_2 \cdot 1 - p_3 \cdot 1 = 0 & p_1 + p_2 = p_3 \end{array}$$

$p_1 + p_2 + p_3 = 1$

$\Rightarrow p_1 = \frac{1}{4}, \quad p_2 = \frac{1}{4}, \quad p_3 = \frac{2}{4}$

or from balance equation:

$$\left.\begin{array}{l} p_1 \cdot 1 = p_2 \cdot 1 \\ p_2 1 + p_1 1 = p_3 1 \\ p_1 + p_2 + p_3 = 1 \end{array}\right\} \quad p_1 = \frac{1}{4} \quad p_2 = \frac{1}{4} \quad p_3 = \frac{2}{4}$$

c.) T_1 - holding time in the state S_1

$$T_1 = \min(\underset{\underset{\underset{Exp(1)}{\parallel}}{Exp(q_{12})}}{\underset{\uparrow}{S_{1\to 2}}}, \underset{\underset{\underset{Exp(1)}{\parallel}}{Exp(q_{13})}}{\underset{\uparrow}{S_{1\to 3}}})$$

minimum of two Exp. dist. r.v.

$\Rightarrow Exp(\underbrace{q_{12} + q_{13}}_{q_1}) = Exp(2)$

$P(T_1 > 2) = 1 - F(2) = e^{-q_1 \cdot T_1} = e^{-4} \approx 0.018$