Homework 3

Finite Differences and Absolute Stability

due February 7, 23:59

Guidelines:

- Use only a **single pdf**. Preferably scan your homework, if you take a photo, make sure it is sharp and bright;
- **Include all plots in the pdf** in the right place;
- Do not present plots without commenting them: always write a (short) description of what the plot tells you and what you can conclude from it;
- You can work in groups up to 3;
- If you work in groups:
  - write the names of all group members at the beginning of the report;
  - you can work and discuss together but each group member is required to submit an individual report written with his/her own words; **copy-paste reports will not be accepted**
- Please use the following naming convention: "**surname_hwX.pdf**" and include all Matlab files as a single **separate** archive: "**surname_hwX.zip**". X is the number of the homework.

Task 1: Finite Difference Schemes (a)

Find the highest order finite differences approximation possible of the first derivative of $u(x)$ at the grid nodes $x = x_i$ based on four grid values u_{i-1}, u_i, u_{i+1} and u_{i+2}, where $u_i := u(x_i)$. Assume equidistant grid spacing, *i.e.* $\Delta x := x_{i+1} - x_i = x_i - x_{i-1}$, for all i.

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_i} \approx f(u_{i-1}, u_i, u_{i+1}, u_{i+2})$$

a) Give the approximation of the derivative.

b) What is the leading error term? What is the order of this scheme?

Task 2: Finite Difference Schemes (b)

Do as in Task 1: find the highest order finite differences approximation and give the expression of the derivative for:

a) the first derivative of $u(x)$ at grid nodes x_i based on three grid values u_{i-1}, u_i, u_{i+1}:

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_i} \approx f(u_{i-1}, u_i, u_{i+1})$$

what are the leading error term and the order of the scheme in this case?

b) the second derivative of $u(x)$ at grid nodes x_i based on three grid values u_{i-1}, u_i, u_{i+1}:

$$\left.\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right|_{x=x_i} \approx f(u_{i-1}, u_i, u_{i+1})$$

what are the leading error term and the order of the scheme?

Task 3: Stability Criterion

The range of absolute stability of the linear multistep 2^{nd} order Adams–Bashforth method is studied. This time-stepping method for an initial value problem of the form $\mathrm{d}u(t)/\mathrm{d}t = f(u,t),\ u(t_0) = u_0$ is:

$$u^{n+1} = u^n + \frac{\Delta t}{2}\left[3f^n - f^{n-1}\right],$$

where

$$f^n := f(u^n, t^n), \text{ and } t^n := n\Delta t.$$

a) Is this an explicit or implicit method? Why?

b) Consider the Dahlquist equation:

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= f(u,t) = \lambda u, \qquad \lambda \in \mathbb{C},\, t \geq 0, \\ u(0) &= 1. \end{aligned} \tag{1}$$

Derive the amplification factor $G(\lambda\Delta t) = u^{n+1}/u^n$ for the 2^{nd} order Adams–Bashforth representation of equation (1) and plot with MATLAB the stability region $|G(\lambda\Delta t)| \leq 1$ in the complex plane. Also plot the stability region of the Explicit Euler method and compare the two. What can you conclude from the plots? When and why should you use the 2^{nd} order Adams–Bashforth method instead of the Euler Explicit method?

Task 4: The Modified Wavenumber

On an equispaced grid, the finite-difference derivative of a Fourier mode e^{ikx} can be found by multiplying the function value on each node with the so-called modified wavenumber $\tilde{k}(k)$.

To better understand this concept consider a periodic function

$$f(x + pm) = f(x), \quad m \in \mathbb{Z}.$$

where p is the period length. Let $\underline{f}$ be the discrete representation of $f(x)$ on the equidistant grid where $x_j := j\Delta x$, $\Delta x := p/N$, $j = 0, 1, \ldots, N-1$,

$$\underline{f} := [f_0, f_1, \ldots, f_{N-1}]^\top, \quad f_0 = f_N, \quad \text{where } f_j := f(x_j).$$

For this task consider $p = 2\pi$ and $N = 20$.

a) A first order right-sided finite differences discretization of the derivative $f'(x)$ can be written as

$$\underline{f}'_{num} := [\delta f_0, \delta f_1, \ldots, \delta f_{N-1}]^\top = \underline{\underline{D}}\ \underline{f},$$

where

$$\delta f_j := \frac{f_{j+1} - f_j}{\Delta x}.$$

Use MATLAB to assemble the system matrix $\underline{\underline{D}}$ (remember that $f_0 = f_N$). Include $\underline{\underline{D}}$ in the written report.

b) Consider $f(x) = e^{ikx}$ and derive the expression for the modified wavenumber $\tilde{k}$ for the right-sided finite-difference scheme. Non-dimensionalise the wavenumber with the grid spacing, *i.e.* derive the expression for $\tilde{k}\Delta x$.

c) From now on assume that $k = 5$ (*i.e.* consider a specific wave). Compute the derivative in a discrete (δf_j) and analytical ($f'(x)|_{x=x_j}$) manner at every grid point. Use the previously defined $\underline{\underline{D}}$ for the discrete derivative. Plot the real part for both the numerical and the analytical derivative as a function of x.