SF2722 Differential Geometry, spring 2017
Homework 2

Examiner: Mattias Dahl, Hans Ringström.
Code of conduct (Hederskodex): It is assumed that:

- you shall solve the problems on your own and write down your own solution,
- if you in spite of this are using something you have gotten from somewhere else for some reason (a friend, a book or the internet etc.) you should give a reference to the source.

Deadline: Hand in your solutions before Tuesday February 7 (we will not accept solutions handed in after Tuesday February 14).

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1. Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$, where

$$\tilde{x} = x, \qquad \tilde{y} = y + x^3.$$

Let p be the point $(1, 0) \in \mathbb{R}^2$ (in standard coordinates), and show that

$$\frac{\partial}{\partial x}|_p \neq \frac{\partial}{\partial \tilde{x}}|_p,$$

even though the coordinate functions x and $\tilde{x}$ are identically equal.

(Lee Exercise 3.17 on page 65.)

2. As introduced in Example 1.5 (page 6) we have coordinate charts

$$\begin{aligned}
\varphi_1([x^1, x^2, x^3]) &= (u^1, u^2) = (x^2/x^1, x^3/x^1) && \text{on} \quad U_1 = \{x^1 \neq 0\},\\
\varphi_2([x^1, x^2, x^3]) &= (v^1, v^2) = (x^1/x^2, x^3/x^2) && \text{on} \quad U_2 = \{x^2 \neq 0\},\\
\varphi_3([x^1, x^2, x^3]) &= (w^1, w^2) = (x^1/x^3, x^2/x^3) && \text{on} \quad U_3 = \{x^3 \neq 0\},
\end{aligned}$$

covering the real projective plane $\mathbb{RP}^2$. Show that there is a smooth vector field on $\mathbb{RP}^2$ which has the coordinate expression $u^1 \frac{\partial}{\partial u^1} - u^2 \frac{\partial}{\partial u^2}$ in the first coordinate chart above. Which are the coordinate expressions for this vector field in the other charts?

Alternative version (not mentioning "vector fields"): A tangent vector to $\mathbb{RP}^2$ is given by the expression $u^1 \frac{\partial}{\partial u^1} - u^2 \frac{\partial}{\partial u^2}$ in the first coordinate chart above. Find the coordinate expression for this vector in the other charts.

3. Show that the definition of tangent vectors as equivalence classes of curves is equivalent to the definition of tangent vectors as derivations at a point.

(See Lee p. 72 and problem 3-8 on p. 76.)