SF2722 Differential Geometry, spring 2017
Homework 3

Examiner: Mattias Dahl, Hans Ringström.
Code of conduct (Hederskodex): It is assumed that:

- you shall solve the problems on your own and write down your own solution,
- if you in spite of this are using something you have gotten from somewhere else for some reason (a friend, a book or the internet etc.) you should give a reference to the source.

Deadline: Hand in your solutions before Tuesday February 21 (we will not accept solutions handed in after Tuesday February 28).

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1. Define $q : \mathbb{S}^n \to \mathbb{RP}^n$ as the restriction of the quotient map $\pi : \mathbb{R}^{n+1} \to \mathbb{RP}^n$ to $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Show that the map q is a smooth covering map.

(Lee Problem 4-10 on page 96.)

2. Define a map $F : \mathbb{S}^2 \to \mathbb{R}^4$ by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Using the smooth covering map of the previous problem, show that F descends to a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

(Lee Problem 4-13 on page 97.)

3. The quaternions $\mathbb{H}$ are described in Lee Problem 7-22 on page 173.

a) Let $\mathbb{H}^*$ be the Lie group of nonzero quaternions, and let $\mathcal{S} \subset \mathbb{H}^*$ be the set of unit quaternions. Show that $\mathcal{S}$ is a properly embedded Lie subgroup of $\mathbb{H}^*$.

(Using the Pauli matrices one can see that $\mathcal{S}$ is isomorphic to the Lie group $SU(2)$.)

b) Identify $\mathbb{R}^3$ with the imaginary quaternions through $(x, y, z) \mapsto xi + yj + zk$. For $p \in \mathcal{S}$ define a linear transformation T_p of $\mathbb{R}^3$ by $v \mapsto pvp^{-1}$. Show that T_p is an orientation-preserving isometry, so that T_p can be identified with an element of the group $SO(3)$.

c) Let $u \in \mathbb{R}^3$ be a unit length imaginary quaternion and $\theta \in \mathbb{R}$. Describe the isometry T_p for $p = \cos\theta + \sin\theta u$. (Hint: study how T_p acts on an orthonormal basis of the form $\{u, v, uv\}$.)

d) Show that the map $p \mapsto T_p$ is a surjective Lie group homomorphism $T : SU(2) = \mathcal{S} \to SO(3)$. Show that the kernel of T consists of ± 1.

e) Conclude that $SO(3)$ is diffeomorphic to 3-dimensional real projective space $\mathbb{RP}^3$.