Recitation 7

Chapter 7:

$\boxed{7.1}$

$\boxed{7.6}$

Chapter 8:

$\boxed{8.4}$

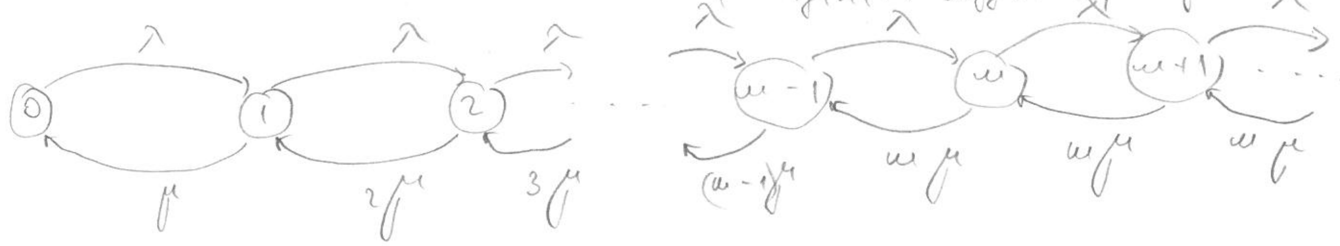
M/M/1

$$N = \frac{\rho}{1-\rho}$$

$$T = \frac{N}{\lambda}$$

M/M/m – Erlang wait systems

- Poisson arrivals (λ)
- Exp service times (μ)
- m identical and independant servers
- ifinite buffer capacity



Stability: $\frac{a}{m} < 1$

$$P(wait) = D_m(a) = \frac{m \cdot E_m(a)}{m - a \cdot (1 - E_m(a))}$$ Erlang C-formula

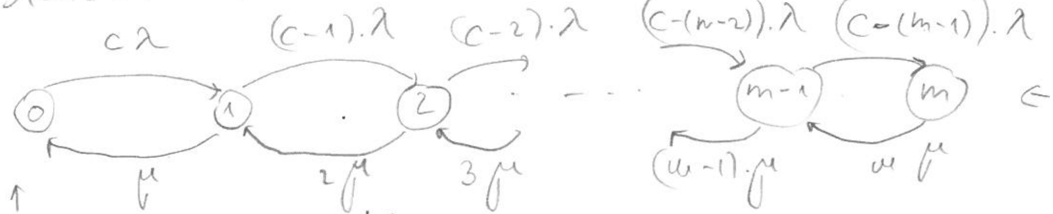
$$\bar{N}_s = \lambda \cdot \bar{x} = a$$

$$\bar{N}_q = \sum_{k=m+1}^{\infty} (k-m) \cdot p_k = \ldots = P(wait) \cdot \frac{a}{m-a}, \quad W = \frac{N_q}{\lambda} \Bigg\} \bar{N}, \bar{T} \ldots$$

$$W = \frac{1}{m\mu - \lambda} \cdot P(wait)$$

M/M/m/m/C

State: # customers under service



← state dependant arrival process

c customers are idle

call blocking (the prob. that an arriving call gets blocked)

$p_k \neq a_k$

time blocking (the proportion of time in blocking state)

$$a_k = \frac{\lambda_k \cdot p_k}{\sum_{i=0}^{m} \lambda_i \cdot p_i}$$

$$\lambda_{eff} = \sum_{i=0}^{m-1} (c-i) \cdot \lambda \cdot p_i$$

PASTA does not hold ! (state dependant arrival process)

⇓

customers with Poisson arrivals see the system as if they came into the system at a random instant of time

7.1 Arrivals: Poisson:

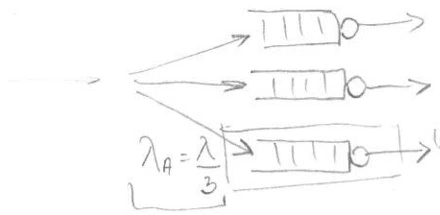
$\lambda = 100 \frac{\text{customers}}{\text{hour}}$

Service: Exp.

3 servers

2 queuing disciplines (A and B)

Case A:

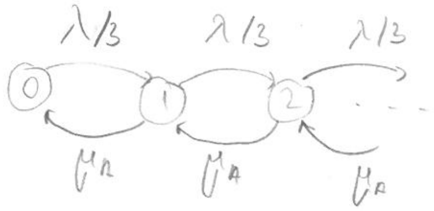


$\lambda_A = \frac{\lambda}{3}$

We can consider only one servers, because the queues are independant $\Rightarrow$ Each queue is $M/M/1$ system

$\bar{x}_A = \frac{1}{\mu_A} = 45\,s$

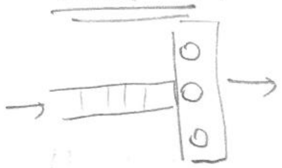
$$\rho_A = \lambda_A \bar{x}_A \Big|_{min} = \frac{\overset{25}{\cancel{100}}}{\cancel{3}\cdot 60} \cdot \frac{3}{\cancel{4}} = \frac{5}{12}$$



$$N = \frac{\rho_A}{1-\rho_A} = \frac{\frac{5}{12}}{\frac{7}{12}} = \frac{5}{7}$$

$$T = \frac{N}{\lambda_A} = \frac{5}{7} \cdot \frac{1}{100/3\cdot 60} = \frac{5}{7} \cdot \frac{3\cdot 60}{100} = \frac{9}{7}\,min = 1,28\,min$$

Case B:

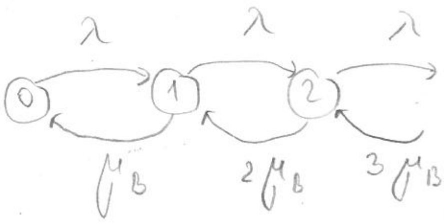


$$\bar{x}_B = \frac{1}{\mu_B} = 50\,s = \frac{5}{6} \frac{\text{cust.}}{\text{min}}$$

$$\lambda_B = \lambda = 100 \frac{\text{cust.}}{h} = \frac{100}{60} = \frac{5}{3} \frac{\text{Cust}}{\text{min}}$$

$$a = \frac{25}{18} \approx 1.38$$

M/M/3



m=3 -servers

Table

$E_m(a) = E_3(1.38) \simeq 0.119$

$D_3(1.38) \approx 0.201$

$$T = \frac{1}{3\cdot\frac{6}{5} - \frac{5}{3}} \cdot 0.201 + \frac{5}{6} = 0.937\,min$$

$T = W + \bar{x}_B$

$$W = \frac{1}{m\cdot\mu_B - \lambda_B} \cdot P(wait)$$

$$P(wait) = D_m(a) = \frac{m \cdot E_m(a)}{m - a\cdot(1-E_m(a))}$$

7.6 Arrivals: Poisson(λ) $\lambda = 3$ ①

Service: Exp(μ) $\mu = 3$

FCFS

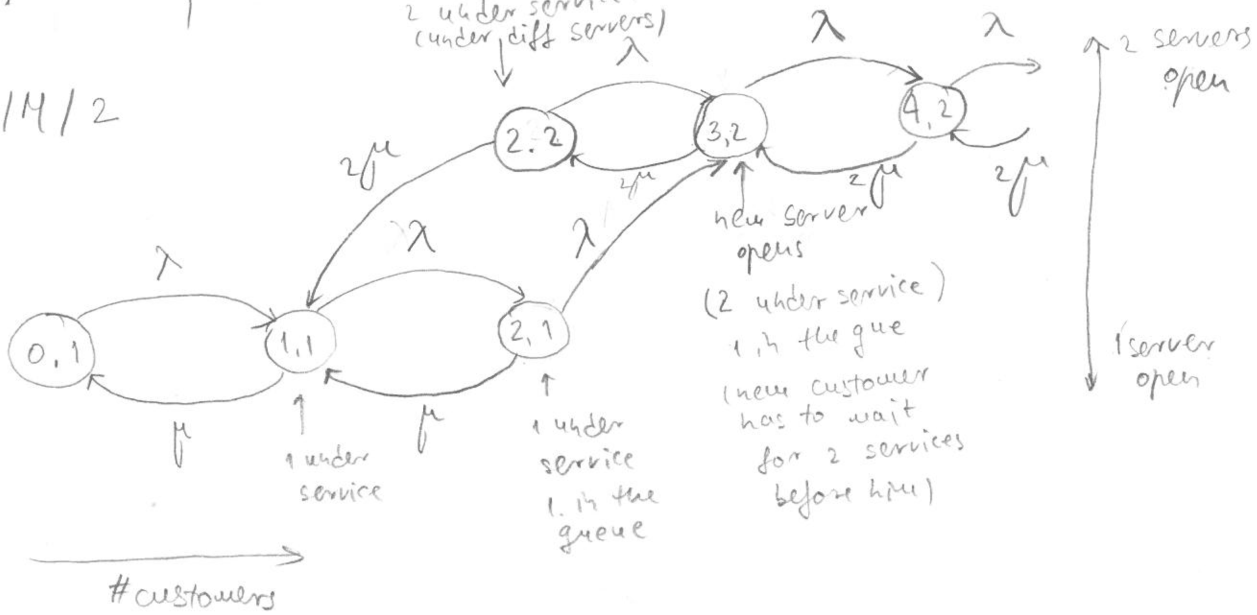
2 servers - one is always available

- the other one starts service when the queue length would become two (and immediately becomes 1, when service starts)

- when the queue becomes empty, the server which becomes idle first is closed (it starts again when the queue length becomes 2)

State: {# customers in the system, # open servers}

a.) M/M/2



b.) at home

c.) $\bar{H}_q = 0 \cdot p_{01} + 0 \cdot p_{11} + 1 \cdot p_{21} + 0 \cdot p_{22} + \sum_{i=3}^{\infty} (i-2) \cdot p_{i2}$

$\bar{H}_S = 0 \cdot p_{01} + 1 \cdot p_{11} + 1 \cdot p_{21} + 2 \cdot p_{22} + \sum_{i=3}^{\infty} 2 \cdot p_{i2}$

d.) $f_W(t \mid i)$, $i \geq 3$

$\Rightarrow$ $i-1$ services have to be completed, each with Exp (2μ)

$\Rightarrow$ sum of $(i-1)$ Exp. dist. independ. var

$\Rightarrow$ Erlang$_{i-1}(2\mu)$

e.) Customer arrives to (1,1): $X_1 \sim$ service if there is a new arrival Exp(λ) or

$f_W(t, (1,1)) \sim$ $X_2 \sim$ service if the previous one completes Exp(μ)

with $(X_1, X_2) \sim$ Exp(λ, μ)

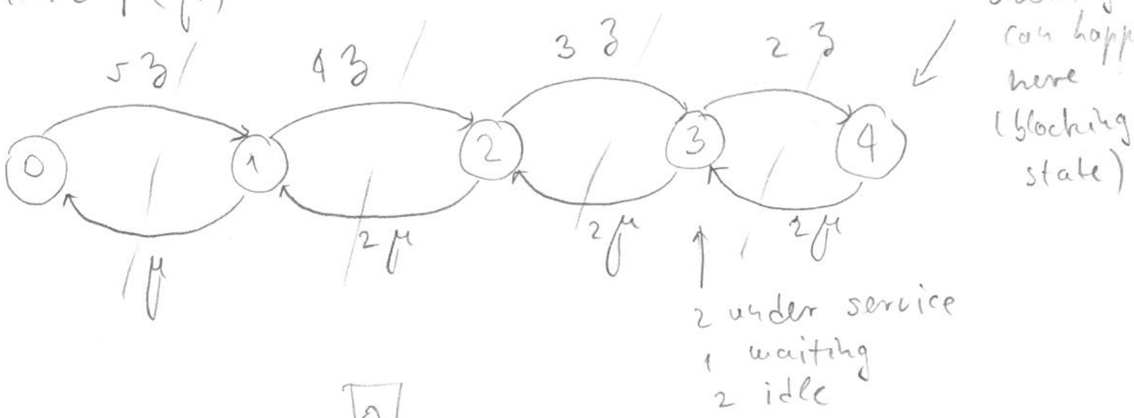
8.4 Markovian system
2 servers
2 queuing positions
5 sources

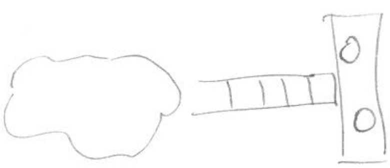
$\left.\begin{array}{l}\text{Markovian system}\\ \text{2 servers}\\ \text{2 queuing positions}\\ \text{5 sources}\end{array}\right\}$ M/M/2/4/5
$\Rightarrow$ can not be modeled with state independant arrivals

(1.)

Idle: $Exp(\zeta)$ ~ thinking time
Service: $Exp(\mu)$

a.)





C = 5

b.) $\mu = \zeta = 1$, $W = ?$

$p_0 \cdot 5\zeta = p_1 \cdot \mu \quad \Rightarrow p_1 = \frac{5\zeta}{\mu} p_0 = 5p_0$

$p_1 \cdot 4\zeta = p_2 \cdot 2\mu \quad \Rightarrow p_2 = \frac{4}{2}\frac{\zeta}{\mu} \cdot p_1 = \frac{5 \cdot 4}{2} p_0 = 10 p_0$

$p_2 \cdot 3\zeta = p_3 \cdot 2\mu \quad \Rightarrow p_3 = \frac{3}{2}\frac{\zeta}{\mu} p_2 = \frac{5 \cdot 4 \cdot 3}{2 \cdot 2} p_0 = 15 p_0$

$p_3 \cdot 2\zeta = p_4 \cdot 2\mu \quad \Rightarrow p_4 = p_3 = 15 p_0$

$p_0 + p_1 + p_2 + p_3 + p_4 = 1$

$\underline{p} = \left\{ \underset{p_0}{\frac{1}{46}}, \underset{p_1}{\frac{5}{46}}, \underset{p_2}{\frac{10}{46}}, \underset{p_3}{\frac{15}{46}}, \underset{p_4}{\frac{15}{46}} \right\}$

$N_q = 0 \cdot p_0 + 0 \cdot p_1 + 0 \cdot p_2 + 1 \cdot p_3 + 2 p_4 = \frac{45}{46}$

$\lambda_{eff} = \sum_{i=0}^{3} \underbrace{(5-i)\cdot\zeta}_{\lambda_i} \cdot p_i = 5\zeta p_0 + 4\zeta p_1 + 3\zeta p_2 + 2\zeta p_3 = \frac{85}{46}$

$W = \frac{N_q}{\lambda_{eff}} = \frac{45}{85}$

c.) Time blocking prob = part of time in blocking state = $p_4 = \frac{15}{46}$ (2.)

d.) Call blocking = the prob that an arriving call gets blocked =

$$= a_4 = \frac{\lambda_4 \cdot p_4}{\sum_{i=0}^{4} \lambda_i \cdot p_i} \approx 0.15$$

e.) P(served without waiting) = $a_0 + a_1 = \frac{\lambda_0 \cdot p_0 + \lambda_1 \cdot p_1}{\sum_{i=0}^{3} \lambda_i \cdot p_i}$

d.) P(served after waiting) = $a_2 + a_3 = \frac{\lambda_2 \cdot p_2 + \lambda_3 \cdot p_3}{\sum_{i=0}^{3} \lambda_i \cdot p_i}$