Recitation 8

Chapter 12

$\boxed{12.2}$

$\boxed{12.3}$

Exam 2009, Dec. Problem 2

M/M/1

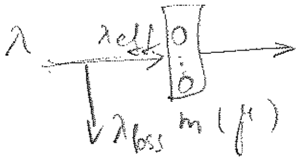
$$\rho = \frac{\lambda}{\mu} = \lambda \cdot \bar{x}$$

$$N = \frac{\rho}{1-\rho}$$

$$T = \frac{N}{\lambda} = \frac{\bar{x}}{1-\rho}$$

$$p_k = (1-\rho)\cdot\rho^k$$

M/M/m/m – E-loss systems



$$\lambda = \lambda_{eff} + \lambda_{loss}$$

$$P(loss) = p_m$$

$$T = T_s = \bar{x}_s$$

Exp. distrib: $X \sim E$

$$F(x) = P(X \le x) = 1 - e^{-ax}$$

$$f_X^*(s) = \int_0^{\infty} e^{-sx} f(x)\, dx = \dots = \frac{a}{a+s}$$

Laplace transform

$$E[X] = -\frac{d}{ds} f_X^*(s)\Big|_{s=0} = \frac{1}{a}$$

$$E[X^2] = \frac{d^2}{ds^2} f_X^*(s)\Big|_{s=0} = \frac{2}{a^2}$$

$$V[X] = E[X^2] - E^2[X] = \frac{1}{a^2}$$

Erlang-n distrib:

$X = X_1 + \dots X_n$ $\quad X_i \sim Exp(a)$ (i.i.d)

$$f^*(s) = \left(\frac{a}{a+s}\right)^n$$

$$f(x) = \frac{(\lambda x)^{n-1}}{(n-1)!}\cdot\lambda\cdot e^{-\lambda x}$$

Poisson distrib:

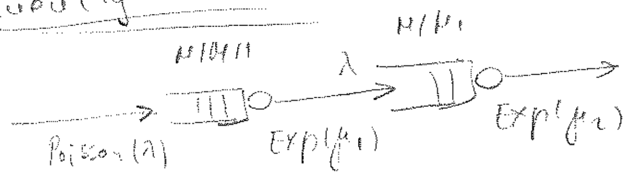
$$\boxed{p_k = P(X=k) = \frac{a^i}{i!} e^{-a}}$$

Geometric distr:

distrib. of the number of attempts needed to get one success (prob. of success – p)

$$X \sim Geo(p) \Rightarrow E[X] = \frac{1}{p}$$

Queuing networks:



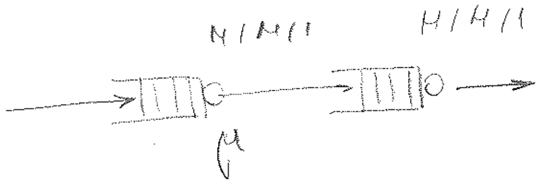
What is the departure process from queue 1?
$\Rightarrow$ Poisson(λ)

Holds for M/M/m, but not for systems with losses and M/G/m systems

The mean number of nodes a customer visits before leaving:

$$\frac{\{\text{Sum arrival intensity to the queues}\}}{\{\text{arrival intensity to the network}\}}$$

$\boxed{12.2}$



$\mu = \mu_1 = \mu_2 = c \cdot \eta$

a.) $T^*(s),\ \bar{T}(t) = ?$

$T = T_1 + T_2$

$T^*(s) = T_1^*(s) \cdot T_2^*(s) = \left(\frac{\mu - \lambda}{s + \mu - \lambda}\right)^2 \Leftarrow$ Remember from M/M/1:

if customer $k+1$ finds k customers in the system $T_{k+1} = \sum_{i=1}^{k+1} X_i$ ← exp. dist. var.

$T^*(s \mid k \text{ customers in the system}) = \underbrace{\left(\frac{\mu}{s + \mu}\right)}_{\text{Laplace transf. of the Exp. distribution}}^{k+1}$

$P_k = (1 - \rho) \cdot \rho^k, \quad \rho = \frac{\lambda}{\mu}$

$T_i^*(s) = \sum_{k=0}^{\infty} (1 - \rho) \cdot \rho^k \cdot \left(\frac{\mu}{s + \mu}\right)^{k+1} = (1 - \rho) \frac{\mu}{s + \mu} \sum_{k=0}^{\infty} \left(\frac{\lambda}{s + \mu}\right)^k =$

$= (1 - \rho) \cdot \frac{\mu}{s + \mu} \cdot \frac{1}{1 - \frac{\lambda}{s + \mu}} = \frac{\mu - \lambda}{s + \mu - \lambda} \to \mathrm{Exp}(\mu - \lambda)$

$f_T(t) = (\mu - \lambda)^2 \cdot t \cdot e^{-(\mu - \lambda) \cdot t}$

↑ sum of two Exp. distrib. var $\Rightarrow$ Erlang-2

b.) $E[T],\ V[T],\ E[T^2] = ?$

$$\left.\begin{aligned} E[T] &= E[T_1] + E[T_2] = 2 \cdot E[T_1] = \frac{2}{\mu - \lambda} \\ V[T] &= V[T_1] + V[T_2] = 2V[T_1] = \frac{2}{(\mu - \lambda)^2} \end{aligned}\right\} \Rightarrow E[T^2] = V[T] + E[T]^2 = \frac{6}{(\mu - \lambda)^2}$$

c.) $P(T > t) = P($one or two services are not finished within $t) =$

$= P(1$ or $\emptyset$ "arrivals" in a Poisson process that describes services$) =$

$= (\mu - \lambda) t \cdot e^{-(\mu - \lambda) \cdot t} + e^{-(\mu - \lambda) t} = e^{-(\mu - \lambda) t} \cdot [1 + (\mu - \lambda) \cdot t]$

EP2200 Queuing theory and teletraffic systems

Final exam, Thursday December 17, 2009, 14:00-19:00

Available teachers: Ioannis Glaropoulos, 08 790 4251

Allowed help: Calculator, Beta mathematical handbook or similar, provided sheets of queuing theory formulas and Erlang tables

1. Consider a processor that takes care of a service. Assume however that the processor suffers from random failures that can put it out of function. The reparation time of the processor is assumed to be exponentially distributed with a mean of 1 hour. The time period between the point that the processor is repaired until its next failure (*time to failure*) is also exponentially distributed with mean of 10 hours.

a) Draw the state diagram and calculate the percentage of time in steady state that the processor is out of function. (2p)

To increase the reliability of the system we install 3 processors that work in parallel. When only one processor is operating, then its time to failure is exponentially distributed with a mean of 10 hours, as above. When 2 processors are operating, then they share the workload and for both of them the mean time to failure becomes 20 hours. Similarly, when all the 3 processors are operating, their mean time to failure becomes 30 hours. Assume finally that there exist 2 service units, so two processors can simultaneously be under reparation. Consider the system in steady state.

b) Draw the state diagram and calculate the percentage of time the system is unavailable (all the processors are out of order). (3p)

c) Calculate the probability that an arbitrary processor that fails has to wait for its reparation to begin and the mean value of the waiting time. (3p)

d) Compute the mean value of the period between two system failures. (2p)

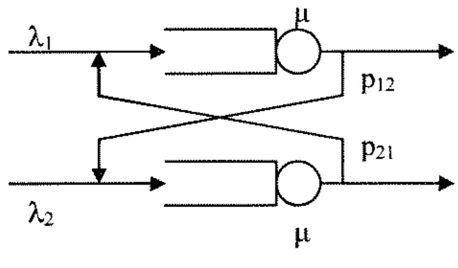
2. Consider the open queuing network on the figure below. The queuing systems in the queuing network have single server, exponential service time with parameter μ=3, and infinite buffer capacity. There are two streams of customers arriving to the queuing network, according to Poisson process. The arrival intensities are $\lambda_1=\lambda_2=1$ customer per second. The probability that after service a customer moves to the other queue is $p_{12}=p_{21}=0.5$. Consider the queuing network in steady state.

a) Calculate the arrival intensities to the two queues of the queuing network. (2p)

b) Calculate the probability that the queuing network is empty at an arbitrary point of time. (3p)

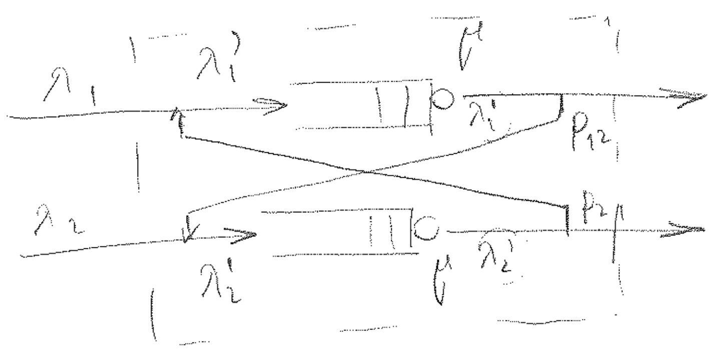
c) Calculate the average time a customer spends in the queuing network. (3p)

d) Calculate the average number of services a customer receives before leaving the network. (2p)



Exam 2009 Dec, Problem 2

2



M/M/1, $\mu = 3$

$\lambda_1 = \lambda_2 = 1 \ \frac{cust}{s}$

$p_{12} = p_{21} = 0.5$

steady state

a.) $\lambda_1' = ?$ $\quad \lambda_1' = \lambda_1 + p_{21} \cdot \lambda_2'$

$\lambda_2' = ?$ $\quad \lambda_2' = \lambda_2 + p_{12} \cdot \lambda_1'$

$$\left.\underbrace{\begin{aligned} \lambda_1' &= \lambda_1 + p_{21}\cdot\lambda_2' \\ \lambda_2' &= \lambda_2 + p_{12}\cdot\lambda_1' \end{aligned}}_{\text{Flow conservation}}\right\} \lambda_1' = \lambda_2' = 2$$

b.) Based on the independency of the queues:

$$P(\text{system empty}) = p_0^1 \cdot p_0^2 = (1-\rho_1)\cdot(1-\rho_2) = \left(1 - \frac{\lambda_1'}{\mu}\right)\cdot\left(1 - \frac{\lambda_2'}{\mu}\right) = \frac{1}{9}$$

c.) Little theorem for the entire queuing network:

$$\bar{T}_{sys} = \frac{\bar{N}}{\lambda_1+\lambda_2} = \frac{\bar{N}_1 + \bar{N}_2}{\lambda_1+\lambda_2} = \frac{4}{2} = 2$$

$$\bar{N}_i = \frac{\rho_i}{1-\rho_i} = 2$$

d.) $$V = \frac{\lambda_1' + \lambda_2'}{\lambda_1 + \lambda_2} = 2$$

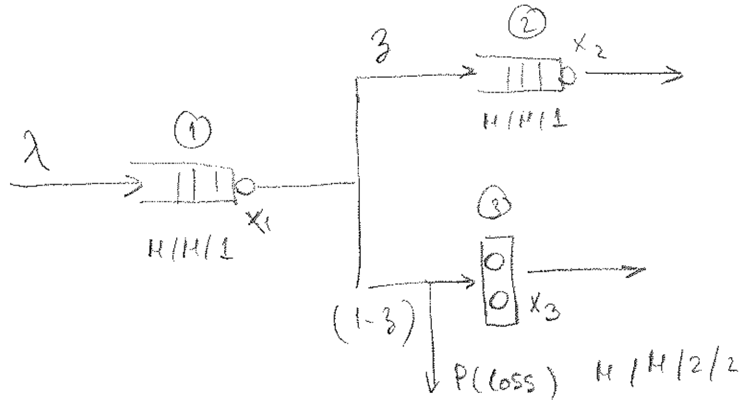
Alt sol:

Number of visits for one job: Geometric $\left(\frac{1}{2}\right)$

$$V = \frac{1}{1/2} = 2$$

Geometric distr: distribution of the number of attempts needed to get one success.

12.3



$\lambda = 4$ cust/min

$x_1 = 10$ s

$x_2 = 30$ s

$x_3 = 20$ s

$3 = 0.2$

1

a.) Arrival rates: $\lambda_1 = \lambda = 4$ cust/min $\Rightarrow \rho_1 = \lambda_1 \cdot x_1 = \frac{4}{60} \cdot 10 = \frac{2}{3}$

$\lambda_2 = 3 \cdot \lambda = 0.8 = \frac{4}{5}$ cust/min $\Rightarrow \rho_2 = \lambda_2 \cdot x_2 = \frac{4}{5 \cdot 60} \cdot 30 = \frac{2}{5}$

$\lambda_3 = (1-3) \cdot \lambda = 3.2 = \frac{16}{5}$ cust/min $\Rightarrow \rho_3 = \lambda_3 \cdot x_3 = 1.067$

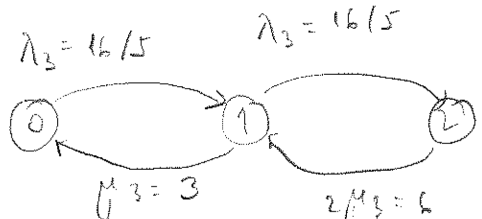
b.) Mean numb. of customers at 1 and 2 $\Rightarrow$ M/M/1 $\quad N = \frac{\rho}{1-\rho}$

$N_1 = \frac{\rho_1}{1-\rho_1} = 2 \qquad N_2 = \frac{\rho_2}{1-\rho_2} = \frac{2}{3}$

c.) Number of cust. rejected per min (where? - at 3) $\lambda_{loss} = ?$

$\lambda = \lambda_{eff} + \lambda_{loss}$

M/M/2/2



$p_0 \cdot \frac{16}{5} = 3 p_1 \Rightarrow p_1 = \frac{16}{15} p_0$

$p_1 \cdot \frac{16}{5} = 6 \cdot p_2 \Rightarrow p_2 = \frac{16}{30} p_1 = \frac{16 \cdot 16}{15 \cdot 30} p_0$

$$\underbrace{p_0 + p_1 + p_2 = 1}$$

$p_0 = \frac{225}{593} \qquad p_1 = \frac{240}{593} \qquad p_2 = \frac{128}{593}$

$p(loss) = p_2 = \frac{128}{593} \approx 0.215$

$\lambda_{los} = \lambda_3 \cdot p(loss) = \lambda_3 \cdot p_2 \approx 0.67 \frac{cust}{min}$

d.) T of not rejected cust - can not use little

$T = T_1 + P(\text{sent to } 2 \mid \text{not rejected}) \cdot T_2 + P(\text{sent to } 3 \mid \text{if not rejected}) \cdot T_3$ (2.)

$= T_1 + \dfrac{\lambda \cdot 3}{\lambda_3 + \lambda_{eff} \text{ for } 3} \cdot T_2 + \dfrac{\lambda_{eff} \text{ for } 3}{\lambda_3 + \lambda_{eff} \text{ for } 3} \cdot T_3$

$M/M/1 \Rightarrow T_1 = \dfrac{x_1}{1-\rho_1}$

$M/M/1 \Rightarrow T_2 = \dfrac{x_2}{1-\rho_2}$

$M/M/2/2 \Rightarrow T_3 = T_S = X_3$

e.) Mean waiting time of a customer not rejected at 3

$W = W_1 + W_2 = \underbrace{T_1 - X_1}_{W_1} + \boxed{\dfrac{\lambda_3}{\lambda_3 + \lambda_{eff} \text{ for } 3}} \cdot (T_2 - X_2) \approx 29{,}8 \text{ sec}$

↑ prob that sent to 2 if not rejected