Jones calculus for optical system

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Key concepts in the course so far

- What is meant by an electro-magnetic response?
- What characterises temporally and spatially dispersive media?
- What is meant by anisotropy?
- What are the roles of the Hermitian and anti-Hermitian parts of the dielectric tensor?
- The wave-operator, $\Lambda_{ij}$:
  - What is the homogeneous wave equation in terms of $\Lambda_{ij}$?
  - What physics is included in $\Lambda_{ij}$ (e.g. how does the different terms in Ampere’s law come into $\Lambda_{ij}$)?
- What is the difference between the dispersion equation and the dispersion relation?
- What characterises a birefringent media?
- Describe the quarter wave plate.
  - What type of media do you need to make a quarter wave plate, why?
  - How does transform linearly and circularly polarised waves?
Outline

• **Quarter wave plates** – map circular to linear polarisation
• **Jones calculus**; matrix formulation of how wave polarization changes when passing through polarizing component
  – Examples: linear polarizer, quarter wave plate

Next lecture:

• Statistical representation of incoherent/unpolarized waves
• Stokes vector and polarization tensor
  – Poincare sphere
• **Muller calculus**; matrix formulation for the transmission of partially polarized waves
Birefringent media

- Previous lecture we noted that in birefringent crystals:
  - Orientate optical axis in the $z$-direction
    
    \[
    K(\omega) = \begin{pmatrix}
    K_\perp(\omega) & 0 & 0 \\
    0 & K_\perp(\omega) & 0 \\
    0 & 0 & K_\parallel(\omega)
    \end{pmatrix}
    \]
  - Orient $k$ perpendicular to the $y$-axis: $\kappa = (\sin \theta, 0, \cos \theta)$
  - The two modes, O-mode and X-mode, are described by:
    
    \[
    \begin{align*}
    n_O^2 &= K_\perp \\
    n_X^2 &= \frac{K_\perp K_\parallel}{K_\perp \sin^2 \theta + K_\parallel \cos^2 \theta}
    \end{align*}
    \]
    
    \[
    \begin{align*}
    \mathbf{e}_O(k) &= (0, 1, 0) \\
    \mathbf{e}_X(k) &\propto (K_\parallel \cos \theta, 0, K_\perp \sin \theta)
    \end{align*}
    \]
  - thus if $K_\perp > K_\parallel$ then $n_O \geq n_X$
    
    the O-mode has larger phase velocity
The quarter wave plate

**Important use of birefringent media: Quarter wave plates**

- uniaxial crystal; normal in \( z \)-direction
- length/width \( L \) in the \( x \)-direction:

\[
L = \frac{c}{\omega} \frac{\pi / 2}{\sqrt{K_{\parallel}} - \sqrt{K_{\perp}}}
\]

(why this formula is explained later!)

- Let the waves travel in the \( x \)-direction, i.e. \( k \) is in the \( x \)-direction and \( \theta = \pi/2 \)

\[
\begin{align*}
n_O^2 &= K_{\perp} \\
n_X^2 &= K_{\parallel}
\end{align*}
\]

\[
\begin{align*}
e_O(k) &= (0, 1, 0) \\
e_X(k) &= (0, 0, 1)
\end{align*}
\]
Exercise: Modifying polarization in a quarter wave plate (1)

Consider a uniaxial plate with axis in the z-direction. The plate is not spatially dispersive.

Let light pass through the plate with the wave vector perpendicular to the plate. Also let the incoming light be linearly polarised with a 45° angle between the electric field and the optical axis of the plate.

Use dispersion relations and eigenvectors:
\[
\begin{align*}
    n_o^2 &= K_\perp \\
    n_x^2 &= K_\parallel
\end{align*}
\]
\[
\begin{align*}
    \mathbf{e}_o(k) &= (0, 1, 0) \\
    \mathbf{e}_x(k) &= (0, 0, 1)
\end{align*}
\]

a) Express the incoming light as a superposition of the eigenvectors.

b) Describe how the polarisation changes as the wave travels through the plate.

c) For which length of the plate is the outgoing wave circularly polarised?

NOTE: Plates that transforms linear polarisation into circular (ellipical) polarisation are called quarter wave plates.

What happens if the incoming light is circularly polarised?
Exercise: Modifying polarization in a quarter wave plate (2)

- Plane wave ansatz has to match dispersion relation
  - when the wave enters the crystal it will slow down, this corresponds to a change in wave length and wave number, \( k \)

\[
\begin{align*}
k_O &= \frac{\omega n_O}{c} = \frac{\omega}{c} \sqrt{K_\perp}, \\
k_X &= \frac{\omega n_X}{c} = \frac{\omega}{c} \sqrt{K_\parallel}
\end{align*}
\]

- since the O- and X-mode travel at different speeds we write

\[
E(t,x) = \Re \left\{ e_O E_O \exp(ik_O x - i\omega t) + e_X E_X \exp(ik_X x - i\omega t) \right\}
\]

- Assume: a linearly polarized wave enters the crystal

\[
E = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} = (e_O + e_X) \Rightarrow E_O = E_X = 1
\]

\[
E(t,x) = \Re \left\{ e_O \exp(ik_O x - i\omega t) + e_X \exp(ik_X x - i\omega t) \right\}
\]

\[
= e_O \cos(k_O x - \omega t) + e_X \cos(k_O x - \omega t + \Delta k x), \quad \Delta k = k_X - k_O
\]

- the difference in wave number causes the O- and X-mode to drift in and out of phase with each other!
Ex: Modifying wave polarization in a quarter wave plate (3)

- The polarization when the wave exits the crystal at $x=L$

$$E(t,x) = \left[ e_o \cos(k_o L - \omega t) + e_x \cos(k_o L - \omega t + \Delta k L) \right]$$

- The crystal converts the linear polarization, $(1,1,0)$ into circular polarization
  - Cyclic mapping: $(1,1,0) \rightarrow (1,i,0) \rightarrow (1,-1,0) \rightarrow (1,-i,0) \rightarrow (1,1,0)$
  - But $(1,0,0)$ and $(0,1,0)$ are unchanged – why?
  - Called a quarter wave plate; a common component in optical systems
  - But work only at one wave length – adapted for e.g. a specific laser!

- Select plate width:

$$\Rightarrow E(t,x) = \left[ e_o \cos(k_o L - \omega t) - e_x \sin(k_o L - \omega t) \right] \text{ Circular polarisation!}$$

- The crystal converts the linear polarization, $(1,1,0)$ into circular polarization
  - Cyclic mapping: $(1,1,0) \rightarrow (1,i,0) \rightarrow (1,-1,0) \rightarrow (1,-i,0) \rightarrow (1,1,0)$
    
  \hspace{1cm} \text{linear} \hspace{1cm} \text{right hand circ} \hspace{1cm} \text{rotated linear} \hspace{1cm} \text{left hand circ} \hspace{1cm} \text{linear}

- But $(1,0,0)$ and $(0,1,0)$ are unchanged – why?
- Called a quarter wave plate; a common component in optical systems
- But work only at one wave length – adapted for e.g. a specific laser!
- In general, waves propagating in birefringent crystal change polarization back and forth between linear to circular polarization
- Switchable wave plates can be made from liquid crystal
  - angle of polarization can be switched by electric control system
- Similar effect: Faraday effect in magnetoactive media (home assignment)
Optical systems

• In optics, interferometry, polarimetry, etc, there is an interest in following how the wave polarization changes when passing through e.g. an optical system.

• For this purpose two types of calculus have been developed;
  – Jones calculus; only for coherent (polarized) wave
  – Muller calculus; for both coherent, unpolarised and partially polarised

• The wave is given by vectors $E$ and $S$ (defined next lecture)
• The polarizing elements are given by matrices $J$ and $M$

\[
E_{out} = J \cdot E_{in}
\]
\[
S_{out} = M \cdot S_{in}
\]
The polarization of transverse waves

- Let's first introduce a new coordinate system representing vectors in the *transverse plane*, i.e. perpendicular to the \( \textbf{k} \).
  - Construct an orthonormal basis for \( \{ \textbf{e}_1, \textbf{e}_2, \kappa \} \), where \( \kappa = \frac{k}{|k|} \)
  - The transverse plane is then given by \( \{ \textbf{e}_1, \textbf{e}_2 \} \), where:
    \[
    \textbf{e}^\alpha = e^\alpha_i \textbf{e}_i
    \]
    where \( \alpha = 1, 2 \) and \( \textbf{e}_i, i = 1, 2, 3 \) is any basis for \( \mathbb{R}^3 \)
  - Ex: \( \textbf{e}^1 = \textbf{e}_x \), \( \textbf{e}^2 = \textbf{e}_y \), \( \kappa = \textbf{e}_x \).
    - Convention: denote \( \textbf{e}^1 \) the *horizontal* and \( \textbf{e}^2 \) the *vertical* directions

- The electric field then has different component representations: \( E_i \) (for \( i = 1, 2, 3 \)) and \( E^\alpha \) (for \( \alpha = 1, 2 \))
  \[
  E_i = e^\alpha_i E^\alpha
  \]
  - similar for the polarization vector, \( \textbf{e}_M \)
  \[
  e_{M,i} = e^\alpha_i e^\alpha_M
  \]

The new coordinates provide 2D representations
Some simple Jones Matrixes

• In the new coordinate system the Jones matrix is 2x2:
  – The electric field entering a linear optical component is: \( E_{\text{in}}^\beta \)
  – Exiting the component is then: \( E_{\text{out}}^\alpha = J^{\alpha\beta} E_{\text{in}}^\beta \)
  – Here \( J^{\alpha\beta} \) is the Jones matrix: \( J^{\alpha\beta} = \begin{bmatrix} J^{11} & J^{12} \\ J^{21} & J^{22} \end{bmatrix} \)

• Example: Linear polarizer transmitting Horizontal polarization, \((L,H)\)

\[
J^{\alpha\beta}_{L,H} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E^H \\ E^V \end{bmatrix} = \begin{bmatrix} E^H \\ 0 \end{bmatrix}
\]

• Example: Attenuator transmitting a fraction \(\rho\) of the energy
  – Note: energy \(\sim \varepsilon_0 |E|^2\)

\[
J^{\alpha\beta}_{\text{Att}}(\rho) = \sqrt{\rho} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \sqrt{\rho} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E^H \\ E^V \end{bmatrix} = \sqrt{\rho} \begin{bmatrix} E^H \\ E^V \end{bmatrix}
\]
Jones matrix for a quarter wave plate

- Quarter wave plates are birefringent (have two different refractive index)
  - align the plate such that horizontal / vertical polarization (corresponding to O/X-mode) has wave numbers $k^1 / k^2$
    \[
    \begin{bmatrix}
    E^H(x) \\
    E^V(x)
    \end{bmatrix} = \begin{bmatrix}
    E^H(0) \exp(ik^1 x) \\
    E^V(0) \exp(ik^2 x)
    \end{bmatrix}
    \]
  - let the light enter the plate start at $x=0$ and exit at $x=L$
    \[
    E(L) = \begin{bmatrix}
    e^{ik^1 L} & 0 \\
    0 & e^{ik^2 L}
    \end{bmatrix} \begin{bmatrix}
    E^H(0) \\
    E^V(0)
    \end{bmatrix} = J_{ph} E(0)
    \]
    - where $Ph$ stands for phaser

- Quarter wave plates change the relative phase by $\pi/2$
  \[
  k^1 L - k^2 L = \pm \pi/2 \rightarrow J_Q = e^{ik^1 L} \begin{bmatrix}
  1 & 0 \\
  0 & \pm i
  \end{bmatrix}
  \]
  - usually we considers only relative phase and skip factor $\exp(ik^1 L)$
Jones matrix for a rotated birefringent media

- If a birefringent media (e.g. quarter wave plates) is not aligned with the axis of our coordinate system…
  - …then we may use a rotation matrix

\[
R(\theta) = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix} \rightarrow R^{-1}(\theta) = R(-\theta)
\]

- In the rotated system the plate is a phaser!

\[
\begin{bmatrix}
E_{M1}(x) \\
E_{M2}(x)
\end{bmatrix} = \begin{bmatrix}
e^{ik^1x} & 0 \\
0 & e^{ik^2x}
\end{bmatrix} \begin{bmatrix}
E^1(0) \\
E^2(0)
\end{bmatrix}
\]

- Relation between original and rotated systems

\[
\begin{bmatrix}
E^1(x) \\
E^2(x)
\end{bmatrix} = R(-\theta) \begin{bmatrix}
E_{M1}(x) \\
E_{M2}(x)
\end{bmatrix} = R(-\theta) \begin{bmatrix}
e^{ik^1x} & 0 \\
0 & e^{ik^2x}
\end{bmatrix} \begin{bmatrix}
E^1(0) \\
E^2(0)
\end{bmatrix}
\]

\[
= R(-\theta) \begin{bmatrix}
e^{ik^1x} & 0 \\
0 & e^{ik^2x}
\end{bmatrix} R(\theta) \begin{bmatrix}
E^1(0) \\
E^2(0)
\end{bmatrix}
\]

\[
J^{\alpha\beta}_{Ph} = R(-\theta) \begin{bmatrix}
e^{ik^1x} & 0 \\
0 & e^{ik^2x}
\end{bmatrix} R(\theta)
\]
Weakly anisotropic media

• Consider for weakly anisotropic media where the modes are still transverse:

\[ K^{\alpha\beta} = n_0^2 \delta^{\alpha\beta} + \Delta K^{\alpha\beta} \]

– where \( \Delta K^{\alpha\beta} \) is a small perturbation

• The wave equation

\[
\left( n^2 \delta^{\alpha\beta} - K^{\alpha\beta} \right) E^\beta = 0 \quad \Rightarrow \quad \left( n^2 - n_0^2 \right) E^\alpha = \Delta K^{\alpha\beta} E^\alpha
\]

– when \( \Delta K_{ij} \) is a small, the 1st order dispersion relation reads: \( n^2 \approx n_0^2 \)

– the left hand side can then be expanded to give

\[
n^2 - n_0^2 = (n - n_0)(n + n_0) = (n - n_0)n_0 \left[ 2 + \frac{(n - n_0)}{n_0} \right] \approx 2n_0(n - n_0)
\]

\[
2n_0(n - n_0)E^\alpha \approx \Delta K^{\alpha\beta} E^\alpha
\]
The wave equation in Jones calculus

- Inverse Fourier transform, when $k_0 = \omega n_0 / c$:
  
  $$2k_0 (k - k_0) E^\alpha \approx \frac{\omega^2}{c^2} \Delta K^{\alpha\beta} E^\alpha \iff 2k_0 (-i \frac{\partial}{\partial x} - k_0) E^\alpha \approx \frac{\omega^2}{c^2} \Delta K^{\alpha\beta} E^\alpha$$

- Factor our the eikonal with wave number $k_0$:
  
  $$E^\alpha = E_0^\alpha (x) \exp(ik_0 x)$$

- The wave equation can then be simplified:
  
  $$(-i \frac{\partial}{\partial x} - k_0) E_0^\alpha (x) \exp(ik_0 x) \approx \frac{\omega^2}{2k_0 c^2} \Delta K^{\alpha\beta} E_0^\alpha (x) \exp(ik_0 x)$$

  $$\frac{dE_0^\alpha}{dx} \approx i \frac{\omega}{2n_0 c} \Delta K^{\alpha\beta} E_0^\beta$$

The differential transfer equation in the Jones calculus!

(We will use this relation in the next lecture)
Summary

• Quarter wave plates splits incoming waves in O- and X-mode and...
  – Transforms circular polarisation into linear polarisation
  – Transforms linear polarisation into elliptic polarisation
  – Circular map: \((1,1,0) \rightarrow (1,i,0) \rightarrow (1,-1,0) \rightarrow (1,-i,0) \rightarrow (1,1,0)\)

• Transverse waves have \(\mathbf{E}\); thus use \(\mathbf{E}\)-components in the transverse plane: \(\mathbf{E} = E^1 \mathbf{e}^1 + E^2 \mathbf{e}^2 = E^\alpha \mathbf{e}^\alpha\)
  – Orientation of coordinates: \(\mathbf{e}^1 \times \mathbf{e}^2 = \hat{\mathbf{k}}\); \(\mathbf{e}^1\) - horizontal, \(\mathbf{e}^2\) - vertical

• Optical components can be described by Jones Matrix:
  – Linear polariser
    \[
    J_{L,H}^{\alpha\beta} = \begin{bmatrix}
    1 & 0 \\
    0 & 0
    \end{bmatrix}
    \]
  – Phaser (birefringent media)
    \[
    J_{Ph}^{\alpha\beta}(kx) = \begin{bmatrix}
    1 & 0 \\
    0 & \exp(ikx)
    \end{bmatrix}
    \]
  – Quarter wave plates
    \[
    J_{Q\pm}^{\alpha\beta} = J_{Ph}^{\alpha\beta}(\pm\pi/2) = \begin{bmatrix}
    1 & 0 \\
    0 & \pm i
    \end{bmatrix}
    \]
    Verify that \(J_{Q\pm}^{\alpha\beta}\) reproduces the circular mapping above!