Recitation 9

Chapter 9:

$\boxed{9.1}$

$\boxed{9.2}$

Collection of exam problems:

$\boxed{\text{Problem 6}}$

M/Er/1/C

- the server consists of r stages
- each stage exp. distrib with $r\mu$
- when customer under service leaves the system the next one can come in

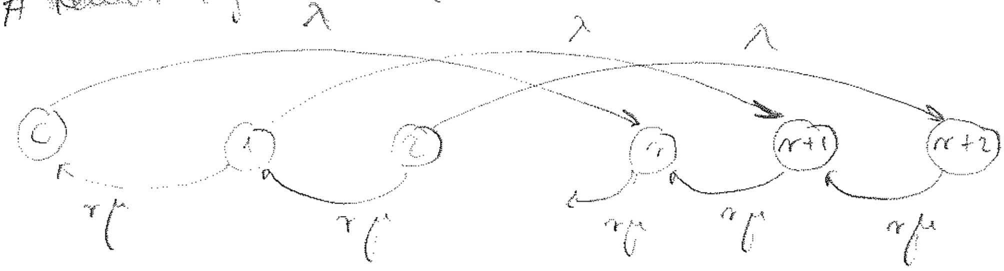
$X_i \sim Exp(r\mu)$

$$E[X] = r \cdot E[X_i] = r \cdot \frac{1}{r\mu} = \frac{1}{\mu}$$

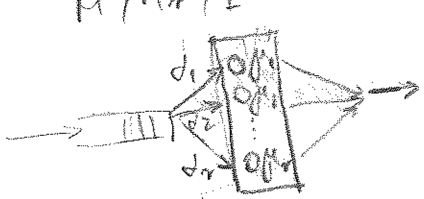
$$V[X] = r \cdot V[X_i] = r \cdot \frac{1}{r^2\mu^2} = \frac{1}{r\mu^2}$$

$$C_x^2 = \frac{V[X]}{E[X]^2} = \frac{1}{r}$$

Remaining service stages:



M/Hr/1



$$\sum_{i=1}^{r} \alpha_i = 1$$

- r exponential servers with different μ-s
- server i is chosen with the prob α_i

$$E[X] = \sum_{i=1}^{r} \alpha_i \cdot E[X_i]$$

$$E[X^2] = \sum_{i=1}^{r} \alpha_i \cdot E[X_i^2]$$

$$V[X] = E[X^2] - E^2[X]$$

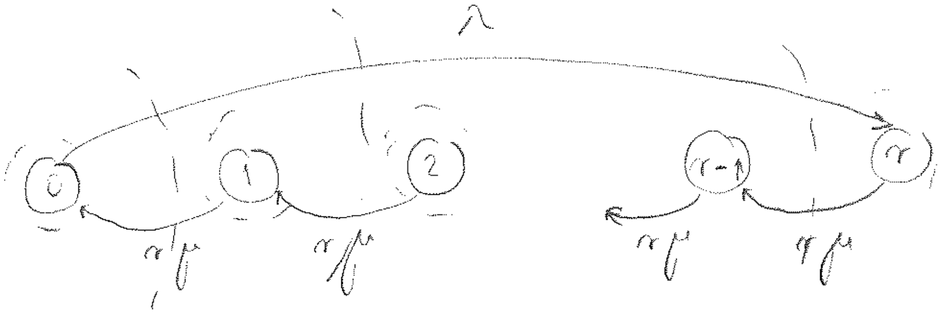
$$C_x^2 = \frac{V[X]}{E[X]^2} = \frac{E[X^2]}{E^2[X]} - 1 \geq 1$$

3.1) $M/E_r/1$ system without buffer

states: the number of stages left of service

Arrival: Poisson (λ)
Service: $E[X] = \frac{1}{\mu}$ $\Bigg\}$ $\rho = \frac{\lambda}{\mu}$

a.)



Global balance equations:

$p_0 \cdot \lambda = p_1 \cdot r\mu \qquad p_1 = \frac{\lambda}{r\mu} p_0$

$p_1 \cdot r\mu = p_2 \cdot r\mu \qquad p_{i+1} = p_i \quad i \geq 1$

$\vdots$

$p_r \cdot r\mu = p_0 \cdot \lambda \qquad p_r = \frac{\lambda}{r\mu} \cdot p_0$

$\sum_{i=0}^{r} p_i = 1 \Rightarrow p_0 \cdot \left(1 + r \cdot \frac{\lambda}{r\mu}\right) = 1$

$p_0 \cdot (1 + \rho) = 1$

$p_0 = \frac{1}{1+\rho} = \frac{\mu}{\mu + \lambda}$

$p_i = \frac{\rho}{r \cdot (1+\rho)}$

Local balance equations:

$p_0 \cdot \lambda = p_1 \cdot r\mu$
$p_0 \cdot \lambda = p_2 \cdot r\mu$
$p_0 \cdot \lambda = p_3 \cdot r\mu$
$\vdots$
$p_0 \cdot \lambda = p_r \cdot r\mu$

$p_i = \frac{\lambda}{r\mu} p_0 \ldots$

$\sum_{i=0}^{r} p_i = 1$

b.) $P(\text{busy}) = 1 - p_0 = 1 - \frac{1}{1+\rho} = \frac{\rho}{1+\rho} = P(\text{blocking})$

Collection of Exam problems 6

$M/E_2/1/2$

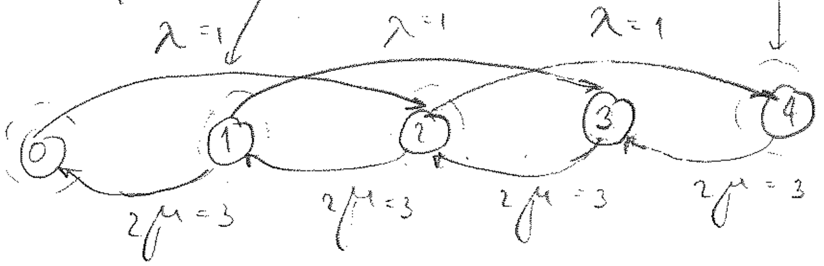
$\lambda = 1 \ \frac{job}{min}$

$\mu = 1.5 \ \frac{job}{min}$

1 customer 1 stage left

1 customer under service with 2 stages left 1 customer in the queue

a.)



$p_0 = 3p_1 \Rightarrow p_1 = \frac{1}{3}p_0$

$4p_1 = 3p_2 \Rightarrow p_2 = \frac{4}{3}p_1 = \frac{4}{9} \cdot p_0$

$4p_2 = p_0 + 3p_3 \Rightarrow p_3 = \frac{4p_2 - p_0}{3} = \frac{7}{27}p_0$

$p_2 = 3p_4 \Rightarrow p_4 = \frac{p_2}{3} = \frac{4}{27}p_0$

$\sum_{i=0}^{4} p_i = 1 \Rightarrow \left(1 + \frac{1}{3} + \frac{4}{9} + \frac{7}{27} + \frac{4}{27}\right) \cdot p_0 = 1 \Rightarrow p_0 = \frac{27}{59},$

$p_1 = \frac{9}{59}, \quad p_2 = \frac{12}{59}, \quad p_3 = \frac{7}{59}, \quad p_4 = \frac{4}{59}$

b.) - $P(\text{blocking}) = p_3 + p_4 = \frac{11}{59}$

- Mean time the system stays in blocking state?

* Blocking state can start in: State 3 $\Rightarrow T(\text{blocking}) = \frac{1}{2\mu}$

State 4 $\Rightarrow T(\text{blocking}) = \frac{1}{2\mu} + \frac{1}{2\mu} = \frac{1}{\mu}$

* What is the probability that the system moves to state 3 or 4:

$\Rightarrow$ arrival in state 1 or 2

$T(\text{block}) = \frac{p_1}{p_1+p_2} \cdot \frac{1}{2\mu} + \frac{p_2}{p_1+p_2} \cdot \frac{1}{\mu} = \frac{2}{3}$ min

- Mean time the system stays in non-blocking state:

$P(\text{blocking}) = \frac{T(\text{blocking})}{T(\text{blocking}) + T(\text{non-blocking})} \Rightarrow \frac{11}{59} = \frac{2/3}{2/3 + T(\text{non-blocking})}$

$\Rightarrow T(\text{non-blocking}) = \ldots = \frac{96}{33}$ min

·c.) : Mean time a job spends in the system:

Job is accepted in states 0, 1, and 2

State 0: $T_0 = \frac{1}{2\mu} + \frac{1}{2\mu} = \frac{1}{\mu}$

State 1: $T_1 = \frac{1}{2\mu} + \frac{1}{2\mu} + \frac{1}{2\mu} = \frac{3}{2\mu}$

State 2: $T_2 = \frac{1}{2\mu} + \frac{1}{2\mu} + \frac{1}{2\mu} + \frac{1}{2\mu} = \frac{4}{2\mu}$

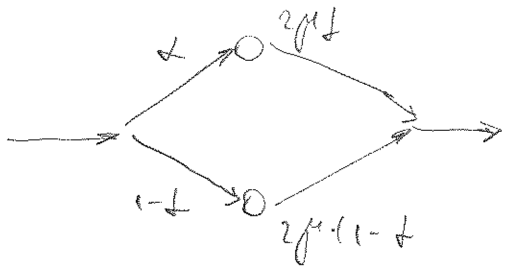
$$T = \frac{\qquad}{P_0 + P_1 + P_2} \cdot \left(P_0 \cdot \frac{1}{\mu} + P_1 \cdot \frac{3}{2\mu} + P_2 \cdot \frac{2}{\mu} \right)$$

9.2 $M/H_2/1$ without buffer

α and $1-\alpha$ probs for server 1 and 2

(1)

$\mu_1 = 2\mu\alpha \; , \; \mu_2 = 2\mu\cdot(1-\alpha)$



a.) Service time distribution (mean, second moment, variance)

$X_1 \sim Exp(2\mu\alpha) \quad E[X_1] = \frac{1}{2\mu\alpha}$

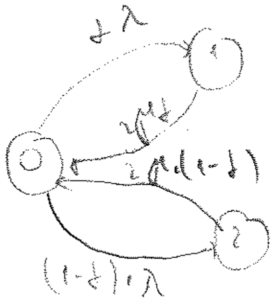
$X_2 = Exp(2\mu\cdot(1-\alpha)) \quad E[X_2] = \frac{1}{2\mu\cdot(1-\alpha)}$

$$C_X \begin{cases} E[X] = \alpha\cdot E[X_1] + (1-\alpha)\cdot E[X_2] = \frac{\alpha}{2\mu\alpha} + \frac{1-\alpha}{2\mu\cdot(1-\alpha)} = \frac{1}{\mu} \\ E[X^2] = \alpha\cdot E[X_1] + (1-\alpha)\cdot E[X_2] = \frac{2\alpha}{4\mu^2\alpha^2} + \frac{2\cdot(1-\alpha)}{4\cdot\mu^2\cdot(1-\alpha)^2} = \\ = \frac{1}{2\mu^2\alpha\cdot(1-\alpha)} \end{cases}$$

$$V[X] = E[X^2] - E[X]^2 = \frac{1}{2\mu^2\cdot\alpha\cdot(1-\alpha)} - \frac{1}{\mu^2} = \frac{1-2\cdot\alpha\cdot(1-\alpha)}{2\mu^2\cdot\alpha\cdot(1-\alpha)}$$

$$C_X = \frac{1-2\alpha\cdot(1-\alpha)}{2\alpha\cdot(1-\alpha)}$$

b.) Steady state, P(idle), P(busy), P(server 1 busy):



$p_0\cdot\alpha\lambda = p_1\cdot 2\mu\alpha \;\Rightarrow\; p_1 = \frac{\lambda}{2\mu}p_0$

$p_0\cdot(1-\alpha)\cdot\lambda = p_2\cdot 2\mu(1-\alpha) \;\Rightarrow\; p_2 = \frac{\lambda}{2\mu}\cdot p_0$

$\Rightarrow 1 = p_0\cdot\left(1+\frac{\lambda}{\mu}\right) \Rightarrow p_0 = \frac{1}{1+\rho} \; , \; p_1 = p_2 = \frac{\rho}{2\cdot(1+\rho)}$

- P(idle) = p_0 ; P(busy) = $1-p_0$; P(server 1 used) = p_1

c.) $\bar{N} = ?$ $\quad \bar{N} = 0\cdot p_0 + 1\cdot(p_1+p_2) = \frac{\rho}{1+\rho}$