Recitation 10

Chapter 10:

$\boxed{10.1}$ a.) b.) + home c.)

$\boxed{10.2}$ a.) b.) at home

$\boxed{10.4}$ a.) b.) c.)

Collection of exam problems:

Problem $\boxed{9}$ a.) b.) c.)

$X \sim Exp(\lambda)$

$f(x) = \lambda \cdot e^{-\lambda x}$

$E[X] = \frac{1}{\lambda}$; $E[X^2] = \frac{2}{\lambda^2}$

$C_X^2 = 1$

$X \sim Erlang\text{-}n(\mu)$

$E[X] = \frac{n}{\mu}$; $V[X] = \frac{n}{\mu^2}$

$C_X^2 = \frac{1}{n}$

$X \sim Geom(p)$

$P(X=k) = (1-p)^{k-1} \cdot p$

$\uparrow$ pmf

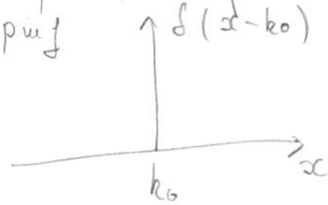
$E[X] = \frac{1}{p}$; $V[X] = \frac{1-p}{p^2}$

Deterministic (degenerate) distrib:

- localized at a point k_0:

$$P(X=k_0) = \begin{cases} 1, & x = k_0 \\ 0, & x \neq k_0 \end{cases} = \delta(x-k_0)$$

$\uparrow$ pmf



$$\mathcal{L}(\delta(x-k_0)) = \int_0^{\infty} e^{-sx} \cdot \delta(x-k_0) \cdot dx = e^{-k_0 x}$$

M/M/1: $N = \frac{\rho}{1-\rho}$, $\bar{N}_S = \rho$

M/Er/1: $E[X] = \frac{1}{\mu}$

Laplace transform:

$$f_X^*(s) = \int_0^{\infty} e^{-sx} f(x) \cdot dx \quad [E[e^{-sx}]]$$

$X \sim Exp(\lambda)$:

$$f_X^*(s) = \frac{\lambda}{\lambda+s}$$

utilization $\triangleq \frac{\lambda_{eff} E[X]}{m}$

$$C_X^2 = \frac{V[X]}{E[X]^2}$$

M/G/1 systems:

Expected waiting time:

$$W = \frac{\lambda \cdot E[X^2]}{2 \cdot (1-\rho)} = \frac{\rho \cdot E[X]}{2 \cdot (1-\rho)} \cdot (1 + C_X^2)$$

$\Rightarrow T, N, N_q$ using Little's

$\mathcal{L}$-transform of the waiting time distribution:

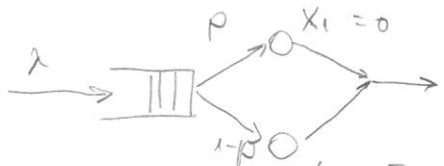
$$W^*(s) = \frac{s \cdot (1-\rho)}{s - \lambda + \lambda \cdot B^*(s)},$$

$B^*(s)$ - $\mathcal{L}$-transform of the service time distribution

Number of customers in the system:

$$Q(z) = B^*(\lambda - \lambda z) \cdot \frac{(1-\rho) \cdot (1-z)}{B^*(\lambda-\lambda z) - z}$$

10.1) M/G/1 system



$E[X_1] = 0, \ E[X_1^2] = 0, \ V[X_1] = 0$

$X_2 \sim Exp(p) \quad E[X_2] = \frac{1}{p}, \ E[X_2^2] = \frac{2}{p^2}, \ V[X_2] = \frac{1}{p^2}$

a.) $b(x) = p \cdot \delta(x) + (1-p) \cdot p \cdot e^{-px}$

$E[X] = p \cdot E[X_1] + (1-p) \cdot E[X_2] = 0 + (1-p) \cdot \frac{1}{p} = \frac{1-p}{p}$

$E[X^2] = p \cdot E[X_1^2] + (1-p) \cdot E[X_2^2] = \frac{2 \cdot (1-p)}{p^2}$

$V[X^2] = E[X^2] - E[X]^2 = \frac{1-p^2}{p^2}$

b.) $W = \frac{\lambda \cdot E[X^2]}{2 \cdot (1-\rho)} = \frac{\lambda \cdot 2 \cdot \frac{(1-p)}{p^2}}{2 \cdot \left(1 - \lambda \cdot \frac{(1-p)}{p}\right)} = \frac{\rho}{p \cdot (1-\rho)}, \quad \rho = \frac{\lambda \cdot (1-p)}{p}$

$B^*(s) \triangleq \int_0^{\infty} e^{-sx} \cdot b(x)\, dx = E[e^{-sx}] = p \cdot E[e^{-s0}] + (1-p) \cdot \underbrace{E[e^{-sx_2}]}_{X_2 \sim Exp(p)} =$

$= p + (1-p) \cdot \frac{p}{p+s} = \dots = \frac{p \cdot (1+s)}{p+s}$

$W^*(s) = \frac{s \cdot (1-\rho)}{s - \lambda + \lambda \cdot B^*(s)} = \dots = \frac{(1-\rho) \cdot (s+p)}{s + p \cdot (1-\rho)} = \dots = (1-\rho) + \rho \cdot \frac{p(1-\rho)}{s + p \cdot (1-\rho)}$

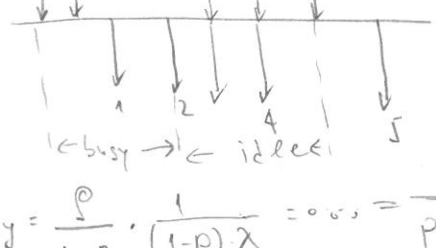
c.) utilization, λ_{max}, $\overline{\tau}_{idle}$, $\overline{\tau}_{busy}$

$\text{utilization} \triangleq \frac{\lambda_{eff} E[X]}{m} = p \cdot \lambda \cdot E[X_1] + (1-p) \cdot \lambda \cdot E[X_2] = \lambda \cdot [p \cdot E[X_1] + (1-p) E[X_2]] =$
$= \lambda \cdot E[X] = \rho = \lambda \cdot \frac{1-p}{p}$

$\lambda_{max}: \ \rho < 1 \Rightarrow \lambda_{max} < \frac{p}{1-p}$

Idle and busy period lenghts:

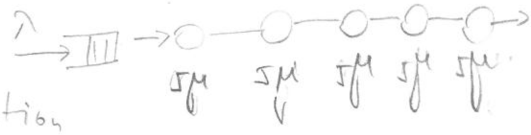
- Only type 2 packets (assigned to server 2) move the system to busy state.
- Arrival of type 2: Poisson$((1-p)\lambda)$



$\overline{\tau}_{idle} = \frac{1}{(1-p)\lambda} \quad \rho = \frac{\overline{\tau}_{busy}}{\overline{\tau}_{busy} + \overline{\tau}_{idle}} \Rightarrow \overline{\tau}_{busy} = \frac{\rho}{1-\rho} \cdot \frac{1}{(1-p) \cdot \lambda} = \dots = \frac{1}{p - \lambda \cdot (1-p)}$

10.2 $\quad$ (1.)

M/M/1 $\quad \bar{N}_{q_{M/M/1}} = 8.1$ packets $\quad X \sim Exp(\mu) \Rightarrow C_x^2 = 1$



a.) $\bar{N}_{M/E_5/1} = ?$ in M/E5/1 with the same utilization

$$\bar{N}_q = \bar{N} - \bar{N}_s = \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho} \Rightarrow \rho = 0.9$$

$$\bar{N}_q = \lambda \cdot W = \lambda \cdot \frac{\rho \cdot E[X]}{2\cdot(1-\rho)} \cdot (1 + C_x^2) = \frac{\rho^2}{2\cdot(1-\rho)} \cdot (1 + C_x^2)$$

$$C_x^2 = \frac{V[X]}{E[X]^2} = \frac{\frac{5}{\cancel{\mu^2}}}{\frac{25}{\cancel{\mu^2}}} = \frac{1}{5}$$

$$V[X] = \frac{r}{\mu^2} = \frac{5}{\mu^2} \quad E[X] = \frac{r}{\mu} = \frac{5}{\mu}$$

$X \sim E_5(\mu)$

$$\frac{\bar{N}_{q_{M/E_5/1}}}{\bar{N}_{q_{M/M/1}}} = \frac{1 + \frac{1}{5}}{1+1} = \frac{3}{5}$$

$$\bar{N}_{q_{M/E_5/1}} = \frac{3}{5} \cdot \bar{N}_{q_{M/M/1}} = 4{,}86 \text{ packets}$$

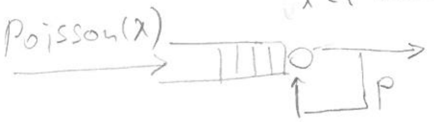
$$\bar{N}_{M/E_5/1} = \bar{N}_s + \bar{N}_{q_{M/E_5/1}} = \rho + \frac{\rho^2}{2\cdot(1-\rho)} \cdot (1 + C_x^2) = 5.76 \text{ packets}$$

$$r \nearrow \infty \Rightarrow C_x^2 = \frac{1}{r} \nearrow 0 \Rightarrow M/E_m/1 \Rightarrow M/D/1$$

b.) at home

10.4

(1.)



$\bar{x} = 1$ (General distribution)

sequence of tx attempts, each $\tau = 1$

a.) service time distribution?

$P(\text{service time} = n \text{ units}) = P(n \text{ transmission until success}) =$

$= p^{n-1} \cdot (1-p) \Rightarrow N \sim \text{Geometric}(1-p)$

b.) message delay = retransmissions + waiting time

System: $M/G/1$

$$E[X] = \sum_{n=1}^{\infty} n \cdot p^{n-1} \cdot (1-p) = (1-p) \cdot \sum_{n=1}^{\infty} (p^n)' = (1-p) \cdot \left[\sum_{n=1}^{\infty} p^n \right]' =$$

$$= (1-p) \cdot \left(\frac{p}{1-p} \right)' = (1-p) \cdot \frac{(1-p) \cdot p}{(1-p)^2} = \frac{1}{1-p}$$

$$E[X^2] = \sum_{n=1}^{\infty} n^2 \cdot p^{n-1} \cdot (1-p) = \dots = \frac{1+p}{(1-p)^2}$$ [or from: $V[X] = \frac{p}{(1-p)^2}$

$$T = E[X] + \underbrace{\frac{\lambda \cdot E[X^2]}{2 \cdot (1-\rho)}} = \frac{1}{1-p} + \frac{\lambda \cdot \frac{1+p}{(1-p)^2}}{2 \cdot \left(1 - \lambda \cdot \frac{1}{1-p}\right)} = \dots$$

W from formula

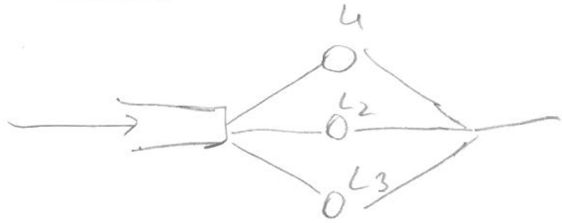
c.) λ_{max}; $\rho < 1$

$$\lambda \cdot E[X] = \lambda \cdot \frac{1}{1-p} < 1$$

$$\boxed{\lambda < 1-p}$$

Collection of the Exam problems, Problem 9

$\equiv$



$\lambda = 10^4 \frac{\text{packets}}{s}$

$c = 34 \cdot 10^6 \frac{b}{s}$

Packet size distribution

$$p_1 = P(L = 40\,B) = 0.65$$
$$p_2 = P(L = 595\,B) = 0.2$$
$$p_3 = P(L = 1500\,B) = 0.15$$

deterministic $\Rightarrow M/G/1$

a.) Transmission times for diff L_i:

$$\tau_i = \frac{L_i \cdot 8}{c}, \quad i = 1, 2, 3.$$

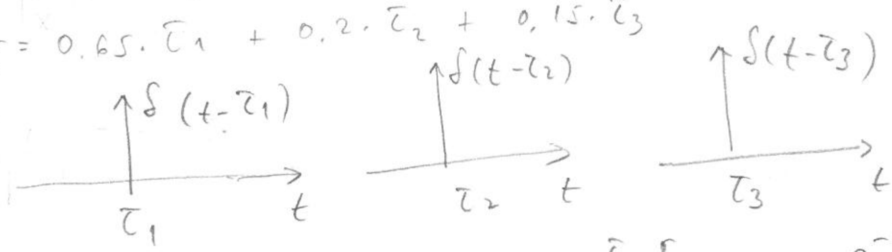
$$E[x] = 0.65 \cdot \tau_1 + 0.2 \cdot \tau_2 + 0.15 \cdot \tau_3$$

$$E[x^2] = 0.65 \cdot \tau_1^2 + 0.2 \cdot \tau_2^2 + 0.15 \cdot \tau_3^2$$

$$W = \frac{\lambda \cdot E[x^2]}{2 \cdot (1-\rho)} = \frac{\lambda \cdot E[x^2]}{2 \cdot (1 - \lambda \cdot E[x])} = \ldots = 0.346\ \text{ms}$$

b.) The L-transform function for the service time distribution:

$$\tau = 0.65 \cdot \tau_1 + 0.2 \cdot \tau_2 + 0.15 \cdot \tau_3$$



$$\mathcal{L}\{\tau\} = B^*(s) = \underline{0.65 \cdot e^{-\tau_1 \cdot s} + 0.2 \cdot e^{-\tau_2 \cdot s} + 0.15 \cdot e^{-\tau_3 \cdot s}}$$

$$B^*(\lambda - \lambda \cdot z) = 0.65 \cdot e^{-\tau_1 \cdot (\lambda - \lambda z)} + 0.2 \cdot e^{-\tau_2 \cdot (\lambda - \lambda z)} + 0.15 \cdot e^{-\tau_3 \cdot (\lambda - \lambda z)}$$

$$\rho = \lambda \cdot E[x] = 0.87$$

$$Q(z) = B^*(\lambda - \lambda \cdot z) \cdot \frac{(1-\rho) \cdot (1-z)}{B^*(\lambda - \lambda z) - z}$$

a.) Packet lengths Exp

(2.)

$\bar{L}_1 = 40B$ with prob p_1

$\bar{L}_2 = 595B$ with prob p_2

$\bar{L}_3 = 1500B$ with prob p_3

$\Rightarrow \tau_i \sim Exp$

$$\bar{\tau}_i = \frac{\bar{L}_i \cdot 8}{C}$$

$$\bar{X} = E[X] = 0.65 \cdot \bar{\tau}_1 + 0.2 \cdot \bar{\tau}_2 + 0.15\,\bar{\tau}_3$$

$$\rho = \lambda \cdot E[X] = 0.87$$

$$T = \frac{\rho}{1-\rho}$$

$$W = T - \bar{X} = 0{,}582 \text{ msec}$$