Recitation 12

Practice problems:

Problem 3

Problem 4

Problem 5

Practice problems

1)

Consider the following two systems. In system A there is one server, the service time is exponentially distributed with parameter $\min(k\mu, m\mu)$, where k is the number of customers in the system and m is a constant. In system B there are m servers and the service intensity of each server is μ. In both cases there is an infinite queue and the arrival process is Poisson, the mean time between arrivals is 1s. $1/\mu$=4s, m=5. Answer the following questions for both systems! (Do not need to calculate state probabilities now, enough if you give parametric solution.)

a) Give the Kendall notation and draw the state diagram of the system.
b) What is the probability of that an arriving customer finds two customers waiting?
c) What is the mean number of customers in the queue?
d) What is the mean time spent in the system?
e) What is the mean time spent in the server?

2)

Consider a local telephone switch with two lines and 1 waiting position. Call arrivals originate from two groups of subscribers, A and B, with five subscribers in each group. Subscribers behave as follows: after finishing a call they are idle for an exponentially distributed time, with a mean of 30min. Then they try to make a new call. If the servers and the queue position are occupied they start a new idle period. Call durations are exponentially distributed with mean of 6 min.

a) Give the Kendall notation and draw the state transition diagram.
b) Calculate the time blocking and call blocking probabilities.
c) Calculate the expected waiting time considering the calls that are waiting for service.
d) Consider now the same system without the waiting position. Assume that subscribers of group B are allowed to make a call only when both lines are free. Draw the state transmission diagram for this case.

3)

Two streams of packets arrive to a multiplexer. Packet sizes are exponentially distributed in stream 1 and Erlang-2 distributed in stream 2. The mean packet size is 500 Bytes in both of the streams. The arrival process is Poisson for both of the streams with intensities 2000 packets per second in stream 1 and 3000 packets per second in stream 2. The link transmission capacity is 32 Mbit/s.

a) Consider the combined stream at the multiplexer. Define the arrival process and give the mean and the variance of the packet size distribution.
b) Assume that there is no buffer at the multiplexer. Give the Kendall notation and the Markov-chain of the system.
c) Calculate the probability that the system is busy.
d) Calculate the probability that an arbitrary packet gets blocked and the probability that an arriving packet of stream 1 gets blocked.

4)

Jobs arrive to a server according to a Poisson process with intensity 2 jobs per time unit. The Laplace transform of the service time distribution is given by $S^*(s)=1/(s+2)+2/(s+4)$. Jobs that arrive when the server is busy are dropped.

a) Give the Kendall notation and draw the state transition diagram for the system.
b) Calculate the utilization of the server.

Assume now that the jobs that arrive when the server is busy are placed in a queue.

c) What is the expected waiting time of the jobs?
d) If 10% of the jobs are given non-pre-emptive priority, what is the expected waiting time of an arbitrary job?
e) Assume that the server has to perform a maintenance procedure every time it finishes the service of a high priority job. It takes 1 time unit on average to perform the maintenance, the duration is exponentially distributed. What is the average waiting time of an arbitrary job in this case?

5)

Consider an open queuing network of three nodes. All the three nodes have infinite buffer capacity, a single server and exponentially distributed service time. Jobs arrive to node 1 and 2 according to a Poisson process. After service at node 1 or 2 jobs enter node 3. After service at node 3 jobs leave the system with probability 0.5 or return to node 2 with probability 0.5. The arrival and service intensities are the followings: $\lambda_1=2$, $\lambda_2=1$, $\mu_1=3$, $\mu_2=6$, $\mu_3=10$.

a, Give the Kendall notation for the three nodes. Motivate your answer.
b, Calculate the arrival intensities at the nodes and the utilization of the nodes.
c, Calculate the probability that the network is empty.
d, Calculate the average number of jobs in the network.
e, Calculate the mean time in the network for an arbitrary job. Calculate the mean time in the network for jobs arriving at node 1 and for jobs arriving at node 2.

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$M / E_2 + M / 1 / 1$ $\qquad C = 32 \cdot 10^6$ Bit/s

Stream 1: $\lambda_1 = 2000 \frac{\text{pkt}}{\text{sec.}}$

$L_1 \sim Exp(\beta_1)$ $\quad E[L] = \frac{1}{\beta_1} = 500$ B

$X_1 \sim Exp(\mu_1)$ $\quad E[X] = \frac{1}{\mu_1} = \frac{500 \cdot 8}{32 \cdot 10^6} = 125 \cdot 10^{-6}$ s $\qquad \mu_1 = 8 \cdot 10^{-3} \frac{\text{packet}}{\text{sec.}}$

Stream 2: $\lambda_2 = 3000 \frac{\text{pkt}}{\text{sec.}}$

$L_2 \sim$ Erlang-2(β_2) $\quad E[L] = \frac{2}{\beta_2} = 500$B

$E[X] = 125 \cdot 10^{-6}$ s $= \frac{2}{\mu_2}$ $\qquad \mu_2 = 16 \cdot 10^{-3} \frac{\text{packet}}{\text{sec.}}$

a) Arrival process $\sim$ Poisson$(\lambda_1 + \lambda_2)$

Packet size distribution

$$E[L] = p_1 E[L_1] + p_2 E[L_2] = \frac{\lambda_1}{\lambda_1+\lambda_2} E[L_1] + \frac{\lambda_2}{\lambda_1+\lambda_2} E[L_2] = 500 \text{ B}$$

Lin. comb.

$$E[L^2] = p_1 E[L_1^2] + p_2 E[L_2^2] = \left[\frac{4}{5} + \frac{3}{5} \cdot \frac{3}{2}\right] \cdot 250\,000 = 425\,000 \text{ B}^2$$

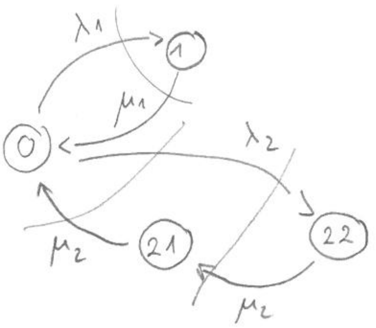
Exp: $\quad E[L_1^2] = \frac{2}{(\beta_1)^2} = 2 \cdot E[L_1]^2 = 50\,00\,00$

Erlang-2 $\quad E[L_2^2] = Var[L_2] + E[L_2]^2 = 2 \cdot Var[L_2^*] + E[L_2]^2 =$

$$= 2 \cdot \frac{1}{\beta_2^2} + \left(\frac{2}{\beta_2}\right)^2 = \frac{6}{\beta_2^2} = \frac{3}{2} \cdot \left(\frac{2}{\beta_2}\right)^2 = \frac{3}{2} \cdot 250\,000$$

b) $M/G/1/1$ or $M/E_2+M/1/1$

c)



$P(\text{system busy}) = 1 - p_0$

$P(\text{packet blocked}) = P(\text{stream 1 packet blocked}) = 1 - p_0$

1.

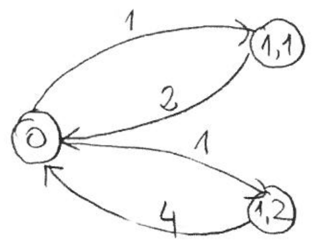
$\lambda = 2$

$$S^*(s) = \frac{1}{s+2} + \frac{2}{s+4} = 0.5 \cdot \frac{2}{s+2} + 0.5 \cdot \frac{4}{s+4} \Rightarrow$$

Service time is <u>hyperexponential</u> :

- $Exp(\mu_1 = 2)$ with probability 0.5
- $Exp(\mu_2 = 4)$ with probability 0.5

a) $M/H_2/1/1$



b)

$$P_{1,1} = \frac{1}{2} P_0 \quad , P_{1,2} = \frac{1}{4} P_0$$

$$P_0 = \frac{1}{1 + \frac{1}{2} + \frac{1}{4}} = \frac{4}{7} \quad , \quad P_{1,1} = \frac{2}{7} \quad , \quad P_{1,2} = \frac{1}{7}$$

$$P_{block} = P_{1,1} + P_{1,2} = 1 - P_0 = \frac{3}{7}$$

$$\text{Utilization} = \rho \cdot (1 - P_{block}) = \lambda \cdot \bar{x}\, (1 - P_{block}) =$$

$$= \underbrace{2 \cdot \left(\tfrac{1}{2} \cdot 0.5 + \tfrac{1}{4} \cdot 0.5\right)}_{\lambda \cdot \bar{x} \,=\, \frac{3}{4}} \left(1 - \tfrac{3}{7}\right) = 0.4286$$

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c) $M/H_2/1$

$$\bar{W} = \frac{\lambda E[S^2]}{2(1-\rho)}$$

$$E[S^2] = S^*(s)'' \Big|_{s=0} = \frac{2}{(s+2)^3} + \frac{4}{(s+4)^3} \Big|_{s=0} = \frac{5}{16}$$

$= p_1 E[S_1^2] + p_2 E[S_2^2] =$ [illegible]

4.2

$$\bar{W} = \frac{2 \cdot \frac{5}{16}}{2\left(1-\frac{3}{4}\right)} = \frac{5}{4}$$

d) Same as in c) (same service times)

e)

$M \sim \mathcal{E}xp(1)$: maintenance time

$S_{hp} = S + M \implies \rho_{hp} = \lambda_{hp}\bar{x}_{hp} = 0.1\lambda\cdot(\bar{x}+1) = 0.275$

$S_{ep} = S \implies \rho_{ep} = \lambda_{ep}\bar{x} = 0.9\lambda\cdot\bar{x} = 0.675$

$$E[S_{hp}^2] = E[(S+M)^2] = E[S^2] + 2E[S]E[M] + E[M^2] = \frac{49}{16}$$

$$E[S_{ep}^2] = E[S^2] = \frac{5}{16}$$

$$\bar{R} = \frac{1}{2}\left(\lambda_{hp}\overline{S_{hp}^2} + \lambda_{ep}\overline{S_{ep}^2}\right) = \frac{1}{2}\left(0.2\cdot\frac{49}{16} + 1.8\,\frac{5}{16}\right) = 0.5875$$

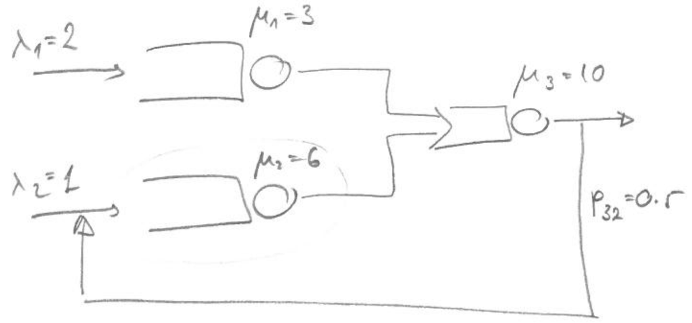
$$\bar{W}_{hp} = \frac{\bar{R}}{1-\rho_{hp}} = \frac{0.5875}{1-0.275} = 0.81$$

$$\bar{W}_{ep} = \frac{\bar{R}}{(1-\rho_{hp})(1-\rho_{hp}-\rho_{ep})} = \frac{0.5875}{(1-0.275)(1-0.275-0.675)} = 16.21$$

$$\bar{W} = 0.1\cdot\bar{W}_{hp} + 0.9\,\bar{W}_{ep} = 14.67$$

(5)

a) M/M/1 → Poisson departure.



b) $\underline{\lambda_1^* = \lambda_1 = 2}$

$$\underline{\lambda_2^*} = \lambda_2 + 0.5\cdot\lambda_3^* \qquad \lambda_2^* = \lambda_2 + 0.5(\lambda_1^* + \lambda_2^*)$$

$$\lambda_3^* = \lambda_1^* + \lambda_2^* \qquad \lambda_2^* = \lambda_2 + 0.5\cdot\lambda_1 + 0.5\cdot\lambda_2^*$$

$$\lambda_2^* = \frac{0.5\cdot\lambda_1 + \lambda_2}{1-0.5} = 2(0.5\lambda_1 + \lambda_2) =$$

$$= 2(1+1) = \underline{4}$$

$$\underline{\lambda_3^*} = 2 + 4 = \underline{6}$$

$$\Rightarrow \rho_1 = \frac{\lambda_1^*}{\mu_1} = \frac{2}{3}, \quad \rho_2 = \frac{\lambda_2^*}{\mu_2} = \frac{4}{6} = \frac{2}{3}, \quad \rho_3 = \frac{\lambda_3^*}{\mu_3} = \frac{6}{10} = \frac{3}{5}$$

Product form

c) $P(\text{Network is empty}) = p_{10}\cdot p_{20}\cdot p_{30} = (1-\rho_1)(1-\rho_2)(1-\rho_3) = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{2}{5} = \frac{2}{45}$

d) $N = N_1 + N_2 + N_3 = \sum_i \frac{\rho_i}{1-\rho_i} = 2 + 2 + \frac{3}{2} = \frac{11}{2} = 5.5$

e) $T = \frac{N}{\lambda_1+\lambda_2} = \frac{11}{2\cdot 3} = \frac{11}{6}$, $T_1 = \frac{N_1}{\lambda_1^*} = \frac{2}{2} = 1$ $\quad T_2 = \frac{N_2}{\lambda_2^*} = \frac{1}{2}$ $\quad T_3 = \frac{1}{4}$

$$T = p_1\overbrace{(T_1 + T_f)}^{T_{S1}} + p_2\overbrace{(T_2+T_f)}^{T_{S2}} = \frac{2}{3}(1+T_f) + \frac{1}{3}\left(\frac{1}{2}+T_f\right) = \frac{2}{3} + \frac{1}{6} + T_f = \frac{5}{6} + T_f = \frac{11}{6}$$

$\Rightarrow \underline{T_f = 1} \quad \Rightarrow T_{S1} = 2 \quad T_{S2} = \frac{3}{2}$

f) $V = \frac{\sum \lambda_i^*}{\sum_i \lambda_i} = \frac{2+4+6}{3} = 4$

Or: With Bernoulli trials ⇒ next page

5. e - other solution

$T_{S1} = T_1 + T_3 + N_f T_{23}$

$T_{S2} = T_2 + T_3 + N_f T_{23}$

N_f: average number of node 2 - nod 3 cycles

T_{23}: time of nod 2-nod 3 cycle:

$T = T_2 + T_3 = \frac{3}{4}$

N_f: each time, leave the network with $p = 1 - p_{32}$, or go back otherwise.

$$N_f = (1-p_{32}) \cdot 0 + (1-p_{32}) p_{32} \cdot 1 + (1-p_{32}) p_{32}^2 \cdot 2 \cdots$$

$$= (1-p_{32}) \cdot \sum_{i=0}^{\infty} i\, p_{32}^{i} = \ldots = \frac{p_{32}}{1-p_{32}} = 1$$

$$T_{S1} = 1 + \frac{1}{4} + 1 \cdot \frac{3}{4} = 2$$

$$T_{S2} = \frac{1}{2} + \frac{1}{4} + \frac{3}{4} = \frac{3}{2}$$