SF2722 Differential Geometry, spring 2017
Homework 5

Examiner: Mattias Dahl, Hans Ringström.

Code of conduct (Hederskodex): It is assumed that:

- you shall solve the problems on your own and write down your own solution,
- if you in spite of this are using something you have gotten from somewhere else for some reason (a friend, a book or the internet etc.) you should give a reference to the source.

Deadline: Hand in your solutions no later than Tuesday April 4 (we will not accept solutions handed in after Tuesday April 11).

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1. Let V be a finite dimensional real vector space. A *bilinear form* on V is a bilinear function from $V \times V$ to $\mathbb{R}$ (in other words, an element of $L(V, V; \mathbb{R})$). A symmetric bilinear form, say b, on V is said to be *non-degenerate* if the following holds: if $b(v, w) = 0$ for all $w \in V$, then $v = 0$. A non-degenerate symmetric bilinear form on V is called a *scalar product* on V. Let g be a scalar product on V and define $\eta : V \to V^*$ by

$$\eta(v)(w) = g(v, w).$$

Prove that η is an isomorphism.

2. Let M be a smooth manifold and let A be a contravariant 2-tensor field on M and B be a covariant 2-tensor field on M. Prove that there is a smooth section of $T^{(1,1)}TM$, say C, with the following property: if A^{ij} and B_{kl} are the components of A and B with respect to local coordinates, then the components of C (with respect to the same coordinates) are given by

$$C^i_j = A^{il} B_{lj},$$

where Einstein's summation convention applies (the construction of C is a special case of a much more general notion of contraction of tensor fields). Let

$$\Delta : \mathfrak{X}(M) \times \mathfrak{X}^*(M) \to C^\infty(M)$$

be defined by $\Delta(X, \omega) = \omega(X)$ (recall that $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M and that $\mathfrak{X}^*(M)$ denotes the set of smooth covector fields on M). Prove that Δ defines a smooth section of $T^{(1,1)}TM$ and compute its components with respect to local coordinates. Finally, let B be a covariant 2-tensor field such that, for each $p \in M$, B induces a scalar product on T_pM. Prove that there is a smooth contravariant 2-tensor field A on M such that if C is constructed from A and B as above, then $C = \Delta$.

3. Let M be a smooth manifold and $V \in \mathfrak{X}(M)$. Show that the Lie derivative operators on covariant tensor fields, $\mathcal{L}_V : \mathcal{T}^k(M) \to \mathcal{T}^k(M)$ for $k \geq 0$, are uniquely characterized by the following properties:

(a) $\mathcal{L}_V$ is linear over $\mathbb{R}$.

(b) $\mathcal{L}_V f = Vf$ for $f \in \mathcal{T}^0(M) = C^\infty(M)$.

(c) $\mathcal{L}_V(A \otimes B) = \mathcal{L}_V(A) \otimes B + A \otimes \mathcal{L}_V(B)$ for $A \in \mathcal{T}^k(M)$ and $B \in \mathcal{T}^l(M)$.

(d) $\mathcal{L}_V(\omega(X)) = (\mathcal{L}_V\omega)(X) + \omega([V, X])$ for $\omega \in \mathcal{T}^1(M)$, $X \in \mathfrak{X}(M)$.

(Lee Problem 12-12 on page 326.)