Chapter 14

The Navier-Stokes equations

14.1 A general continuity equation

We consider the flow of a quantity with density $\phi(x,t)$ at $x \in \Omega \subset \mathbb{R}^n$, with $n = 2, 3$. For a time $t > 0$, the total flow of the quantity through the boundary $\partial\Omega$, is given by

$$\int_{\partial\Omega} \phi u \cdot n \, ds, \tag{14.1}$$

where n is the outward unit normal of $\partial\Omega$, and $u = u(x,t)$ is the velocity of the flow.

For an arbitrary subdomain $\omega \subset \Omega$, the change of the integral of ϕ is equal to the volume source or sink $s = s(x,t)$ minus the total flow of the quantity through the boundary $\partial\omega$,

$$\frac{d}{dt}\int_{\omega} \phi(x,t)\, dx = -\int_{\partial\omega} \phi u \cdot n \, ds + \int_{\omega} s(x,t)\, dx, \tag{14.2}$$

which by Gauss' theorem leads to

$$\int_{\omega} \left(\frac{\partial}{\partial t}\phi(x,t) + \nabla \cdot (\phi u) - s \right) dx = 0, \tag{14.3}$$

for any $\omega \subset \Omega$, and thus we get the general continuity equation

$$\dot{\phi} + \nabla \cdot (\phi u) - s = 0, \tag{14.4}$$

for any $x \in \Omega$, and $t > 0$.

14.2 Mass conservation

We now consider the flow of mass of a continuum, with $\rho = \rho(x,t)$ the mass density at of the continuum. The general continuity equation with $\phi = \rho$, and zero sink $s = 0$, gives the equation for conservation of mass

$$\dot{\rho} + \nabla \cdot (\rho u) = 0. \tag{14.5}$$

A flow is *incompressible* if

$$\nabla \cdot u = 0, \tag{14.6}$$

or equivalently if the *material derivative* is zero,

$$\frac{D\rho}{Dt} = \dot{\rho} + u \cdot \nabla \rho = 0, \tag{14.7}$$

since

$$0 = \dot{\rho} + \nabla \cdot (\rho u) = \frac{D\rho}{Dt} + \rho \nabla \cdot u. \tag{14.8}$$

14.3 Conservation of momentum

Newton's 2nd Law states that the change of *momentum* ρu, is equal to the sum of all forces, including *volume forces*,

$$\int_\omega \rho(x,t) f(x,t)\, dx, \tag{14.9}$$

for a force density $f = (f_1, ..., f_n)$, and *surface forces*,

$$\int_{\partial\omega} n(x,t) \cdot \sigma(x,t)\, ds, \tag{14.10}$$

with the Cauchy stress tensor $\sigma \in \mathbb{R}^{n \times n}$. Gauss' theorem gives the total force as

$$\int_\omega \rho f\, dx + \int_{\partial\omega} n \cdot \sigma\, ds = \int_\omega (\rho f + \nabla \cdot \sigma)\, dx. \tag{14.11}$$

The general continuity equation with $\phi = \rho u$, and the sink given by the sum of all forces, gives the equation for conservation of momentum

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \otimes u) = \rho f + \nabla \cdot \sigma, \tag{14.12}$$

with $u \otimes u = uu^T$, the tensor product of the velocity vector field u. With the help of conservation of mass, we can rewrite the left hand side as

$$\frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho u \otimes u) = u(\dot{\rho} + \nabla \cdot (\rho u)) + \rho(\dot{u} + (u \cdot \nabla) u) = \rho(\dot{u} + (u \cdot \nabla) u),$$

so that we get

$$\rho(\dot{u} + (u \cdot \nabla)u) = \rho f + \nabla \cdot \sigma. \tag{14.13}$$

The Cauchy stress tensor consists of normal stresses on the diagonal, and shear stresses on the off-diagonal. We can decompose σ into a *dynamic pressure*

$$p_d = -\frac{1}{3}\,\mathrm{tr}(\sigma), \tag{14.14}$$

and a *deviatoric stress tensor* $\tau = \sigma + p_d I$, with I the identity matrix,

$$\sigma = -p_d I + \tau, \tag{14.15}$$

so that

$$\rho(\dot{u} + (u \cdot \nabla)u) = \rho f - \nabla p_d + \nabla \cdot \tau. \tag{14.16}$$

The Navier-Stokes equations

We now consider incompressible flow, so that the velocity is divergence free, and we assume the density to be constant. To determine the deviatoric stress we need a constitutive model of the fluid.

For a Newtonian fluid, the deviatoric stress depends linearly on the *strain rate tensor*

$$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{14.17}$$

with $\tau = 2\mu\epsilon$, where μ is the *dynamic viscosity*.

The *incompressible Navier-Stokes equations* takes the form,

$$\dot{u} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = f, \tag{14.18}$$

$$\nabla \cdot u = 0, \tag{14.19}$$

with the *kinematic viscosity* $\nu = \mu/\rho$, and the kinematic pressure $p = p_d/\rho$.

Non-dimensionalization

Solutions to the Navier-Stokes equations may take quite different forms, depending on the balance of the inertial and dissipative terms of the equations. To exhibit this balance, we express the Navier-Stokes equations in terms of the non-dimensional variables u_*, p_*, f_*, x_*, t_*,

$$u = Uu_*, \quad p = Pp_*, \quad x = Lx_*, \quad f = Ff_*, \quad t = Tt_*, \tag{14.20}$$

where U, P, L, T are characteristic scales of the velocity, pressure, force, length and time, respectively. The resulting non-dimensionalized differential operators are scaled as,

$$\frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial t_*}, \quad \nabla = \frac{1}{L}\nabla_*, \quad \Delta = \frac{1}{L^2}\Delta_*, \tag{14.21}$$

which gives

$$\frac{U}{T}\frac{\partial}{\partial t_*}u_* + \frac{U^2}{L}(u_* \cdot \nabla_*)u_* + \frac{P}{L}\nabla_* p_* - \frac{\nu U}{L^2}\Delta_* u_* = F f_*, \tag{14.22}$$

$$\frac{U}{L}\nabla \cdot u_* = 0, \tag{14.23}$$

or,

$$\dot{u} + (u \cdot \nabla)u + \nabla p - Re^{-1}\Delta u = f, \tag{14.24}$$

$$\nabla \cdot u = 0. \tag{14.25}$$

Here we have dropped the non-dimensional notation for simplicity, with

$$T = L/U, \quad P = U^2, \quad F = \frac{U^2}{L}, \quad Re = \frac{UL}{\nu}, \tag{14.26}$$

where the *Reynolds number Re* determines the balance between inertial and viscous characteristics in the flow. For low *Re* linear viscous effects dominate, whereas for high *Re* we have a flow dominated by nonlinear inertial effect, and turbulence for sufficiently high Reynolds number.

In the limit $Re \to \infty$, the viscous term vanished and we are left with the *Euler equations*,

$$\dot{u} + (u \cdot \nabla)u + \nabla p = f, \tag{14.27}$$

$$\nabla \cdot u = 0, \tag{14.28}$$

whereas in the limit $Re \to 0$, we obtain the *Stokes equations* as a model of viscous flow,

$$-\Delta u + \nabla p = f, \tag{14.29}$$

$$\nabla \cdot u = 0, \tag{14.30}$$

with now a different scaling of the pressure and the force,

$$P = \frac{\nu U}{L}, \quad F = \frac{\nu U}{L^2}. \tag{14.31}$$

14.4 Stokes flow

The Stokes equations

The Stokes equations for a domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, and associated normal n, takes the form,

$$-\Delta u + \nabla p = f, \qquad x \in \Omega, \tag{14.32}$$
$$\nabla \cdot u = 0, \qquad x \in \Omega, \tag{14.33}$$
$$u = g_D, \qquad x \in \Gamma_D, \tag{14.34}$$
$$-\nabla u \cdot n + pn = g_N, \qquad x \in \Gamma_N. \tag{14.35}$$

Homogeneous Dirichlet boundary conditions

First assume that we have $\Gamma = \Gamma_D$ and that $g_D = 0$, that is homogeneous Dirichlet boundary conditions for the velocity. We then seek a weak solution to the Stokes equations in the following spaces,

$$V = [H_0^1(\Omega)]^3 \tag{14.36}$$
$$Q = \{q \in L^2(\Omega) : \int_\Omega q\, dx = 0\}, \tag{14.37}$$

where the extra condition on Q is needed to assure uniqueness of the pressure, which otherwise is undetermined up to a constant.

We derive the variational formulation by taking the inner product of the momentum equation with a test function $v \in V$, and the inner product of the continuity equation with a test function $q \in Q$. By Green's formula and the homogeneous Dirichlet boundary condition, we obtain the variational formulation as: find $(u, p) \in V \times Q$, such that,

$$a(u, v) + b(v, p) = (f, v), \tag{14.38}$$
$$b(u, q) = 0, \tag{14.39}$$

for all $(v, q) \in V \times Q$, with

$$a(v, w) = (\nabla v, \nabla w) = \int_\Omega \nabla v : \nabla w\, dx, \tag{14.40}$$
$$b(v, q) = -(\nabla \cdot v, q) = -\int_\Omega (\nabla \cdot v) q\, dx, \tag{14.41}$$

and

$$\nabla v : \nabla w = \sum_{i,j=1}^{3} \frac{\partial v_j}{\partial x_j} \frac{\partial w_j}{\partial x_j}. \tag{14.42}$$

The saddle-point problem

The solution (u, p) to the Stokes equations (14.38-14.39), is also the solution to the constrained minimization problem,

$$\min J(v) = \frac{1}{2}a(v, v) - (f, v) \tag{14.43}$$

under the constraint

$$b(v, q) = 0, \tag{14.44}$$

for which we can formulate the Lagrangian

$$L(v, q) = J(v) + b(v, q), \tag{14.45}$$

so that $p \in Q$ represents a Lagrange multiplier for the constraint $\nabla \cdot u = 0$. The Stokes problem thus represents a saddle-point problem, since

$$L(u, q) \leq L(u, p) \leq L(v, p), \quad \forall (v, q) \in V \times Q. \tag{14.46}$$

Theorem 19. *The saddle-point point problem (14.38-14.39) has a unique solution, if*

(i) the bilinear form a is coercive, i.e. that exists an $\alpha > 0$, such that

$$a(v, v) \geq \alpha \|v\|_V, \tag{14.47}$$

for all $v \in Z = \{v \in V : b(v, q) = 0, \forall q \in Q\}$,

(ii) the bilinear form b satisfies the inf-sup condition, i.e. there exists a $\beta > 0$, such that

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta. \tag{14.48}$$

Mixed finite element approximation

We now formulate a finite element method for solving Stokes equations. Since we use different approximation spaces for the velocity and the pressure, we refer to the method as a mixed finite element method. We seek an approximation $(U, P) \in V_h \times Q_h$, such that,

$$a(U, v) + b(v, P) = (f, v), \tag{14.49}$$
$$b(U, q) = 0, \tag{14.50}$$

for all $(v, q) \in V_h \times Q_h$, where V_h and Q_h are finite element approximation spaces. There exists a unique solution to (14.49-14.50), under certain conditions on the approximation spaces V_h and Q_h.

Theorem 20. *The mixed finite element problem (14.49-14.50) has a unique solution* $(U, P) \in V_h \times Q_h$*, if*

(i) the bilinear form a *is coercive, i.e. that exists an* $\alpha_h > 0$*, such that*

$$a(v, v) \geq \alpha_h \|v\|_V, \tag{14.51}$$

for all $v \in Z_h = \{v \in V_h : b(v, q) = 0, \forall q \in Q_h\}$*,*

(ii) the bilinear form b *satisfies the inf-sup condition, i.e. there exists a* $\beta_h > 0$*, such that*

$$\inf_{q \in Q_h} \sup_{v \in V_h} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geq \beta_h, \tag{14.52}$$

and this unique solution satisfies the following error estimate,

$$\|u - U\|_V + \|p - P\|_Q \leq C \left(\inf_{v \in v_h} \|u - v\| + \inf_{q \in Q_h} \|p - q\| \right), \tag{14.53}$$

for a constant $C > 0$*.*

The pair of approximation spaces must be chosen to satisfy the inf-sup condition, with the velocity space sufficiently rich compared to the pressure space. For example, continuous piecewise quadratic approximation of the velocity and continuous piecewise linear approximation of the pressure, referred to as the Taylor-Hood elements. On the other hand, continuous piecewise linear approximation of both velocity and pressure is not inf-sup stable.

Schur complement methods

Let $V_h = \text{span}\{\phi_j\}_{j=1}^N$ and $Q_h = \text{span}\{\psi_j\}_{j=1}^M$, so that

$$U_k(x) = \sum_{j=1}^N U_k^j \phi_j(x),\ k = 1, 2, 3, \quad P(x) = \sum_{j=1}^M P^j \psi_j(x), \tag{14.54}$$

which leads to the following discrete system in matrix form,

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}. \tag{14.55}$$

The matrix A is invertible, so we can express

$$u = A^{-1}(f - Bp), \tag{14.56}$$

and since $B^T u = 0$,

$$B^T A^{-1} B p = B^T A^{-1} f, \tag{14.57}$$

which is the *Schur complement* equation. If $(B) = \{0\}$, then the matrix $S = B^T A^{-1} B$ is symmetric positive definite.

Schur complement methods take the form

$$p_k = p_{k-1} - C^{-1}(B^T A^{-1} B p_{k-1} - B^T A^{-1} f), \tag{14.58}$$

where C^{-1} is a preconditioner for $S = B^T A^{-1} B$. The Usawa algorithm is based on C^{-1} as a scaled identity matrix, which gives

1. Solve $Au_k = f - Bp_{k-1}$,
2. Set $p_k = p_{k-1} + \alpha B^T u_k$.

Stabilized methods

Approximation spaces of equal order is possible, by stabilization of the standard Galerkin finite element method: find $(U, P) \in V_h \times Q_h$, such that,

$$\begin{aligned} a(U, v) + b(v, P) &= (f, v), & (14.59) \\ b(U, q) + s(P, q) &= 0, & (14.60) \end{aligned}$$

for all $(v, q) \in V_h \times Q_h$, which gives the discrete system

$$\begin{bmatrix} A & B \\ B^T & S \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}. \tag{14.61}$$

For example, the *Brezzi-Pitkäranta* stabilization takes the form,

$$s(P, q) = \int_\Omega h^2 \nabla P \cdot \nabla q \, dx. \tag{14.62}$$