Algorithms and Complexity
2017
Extra Mästarprov 2: Complexity

This test is given to students who failed to get E on the ordinary Mästarprov 2. It consists of two problems. If both problems are solved correctly (basically) the test gives grade E. Your solutions should be handed in latest May 19th 16.00. No collaboration is allowed.

1. Longest Path

The problem LONGEST PATH is the problem of, given an undirected graph G, finding the length of the longest path in G. (The length of the path is the number of edges in the path.). Let us formulate a decision variant of this problem by asking if, given G and a number K as input, there is a path of length $\geq K$. Show that this problem is NP-hard be reducing the problem HAMILTONIAN PATH to LONGEST PATH (the decision variant).

2. Different types of Exact Cover

The problem EXACT COVER can be formulated like this:

Input: A set $M = \{x_1, x_2, ..., x_n\}$. A family $F = \{F_1, F_2, ..., F_m\}$ of subsets of M.

Goal: Is there a subset $\{F_{i_1}, F_{i_2}, ..., F_{i_p}\}$ of F such that every element of M belongs to exactly one of the F_{i_j}:s?

In this problem we contrast this standard formulation with a variant of the problem. We will call this variant FIXED SIZE EXACT COVER (FSEC):

Input: A set $M = \{x_1, x_2, ..., x_n\}$. A family $F = \{F_1, F_2, ..., F_m\}$ of subsets of M. An integer K.

Goal: Is there a subset $\{F_{i_1}, F_{i_2}, ..., F_{i_K}\}$ of F of size K such that every element of M belongs to exactly one of the F_{i_j}:s? (All the F_{i_j}:s are assumed to be distinct.)

We know that EXACT COVER is NP-Complete but we might believe that FSEC is simpler. But, in fact, FSEC is also NP-Complete. We can show this by reducing EXACT COVER to FSEC. One problem with doing this is that it seems impossible to tell what K should be. We cannot tell beforehand what size an EXACT COVER, if there is one, should have. But there is a trick we can use. Given an instance (M, F) of EXACT COVER we can define an instance (M', F', $K = n + 1$) of FSEC. M' will be $M \cup A$ and F' will be $F \cup G$ where A and G are "dummy" sets disjoint from M and F and such that the elements of G are subsets of A. If A and G are chosen in a clever way (M, F) will have an exact cover if and only if (M', F') has a $n + 1$ size cover. How? That is for you to find out. Do this:

1. Show that FSEC is in NP.
2. Find out what A and G should be and by doing that, define the reduction.
3. Prove that the reduction is correct.
4. Show that the reduction can be done in polynomial time.