KTH Teknikvetenskap

SF1624 Algebra och geometri
Exam
Thursday, 8 June 2017

Time: 08:00-11:00
No books/notes/calculators etc. allowed
Examiner: Tilman Bauer

This exam consists of six problems, each worth 6 points.

Part A comprises the first two problems. The bonus points from the seminars will be automatically added to the total score of this part, which however cannot exceed 12 points.

The next two problems constitute part B, and the last two problems part C. The latter is mostly for achieving a high grade.

The thresholds for the respective grades are as follows:

Grade	A	B	C	D	E	Fx
Total sum	27	24	21	18	16	15
of which in part C	6	3	–	–	–	–

To get full score on a problem, your solution must be well-presented and easy to follow. In particular, you should define your notation; clearly explain the logical structure of your argument in words or symbols; and motivate and explain your argument. Solutions severely lacking in these respects will achieve at most 3 points.

PART A

1. Lett $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the map

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x+2y \\ x+y+2z \\ 4x+3y+2z \end{bmatrix}.$$

(a) Find the standard matrix for the map T. **(1 p)**
(b) Find a basis for the nullspace of T. **(3 p)**
(c) Determine the dimension of the range of T. **(2 p)**

2. We are given the matrix

$$A = \begin{bmatrix} 1 & 6 \\ 3 & -2 \end{bmatrix}.$$

(a) Find all eigenvalues and corresponding eigenvectors for A. **(3 p)**
(b) Find a matrix U and a diagonal matrix D such that $A = UDU^{-1}$. **(1 p)**
(c) Compute $A^{123}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. **(2 p)**

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p.t.o.

Part B

3. Consider two lines in $\mathbb{R}^3$: L_1, given by $\begin{bmatrix}1\\1\\1\end{bmatrix} + t\begin{bmatrix}-1\\1\\-1\end{bmatrix}$, and L_2, going through the points $(-1,-1,2)$ and $(1,b,1)$, where b is a constant.

(a) Determine all values of b such that L_1 and L_2 intersect. **(3 p)**

(b) Find an equation of the plane containing L_1 and L_2 for $b = 1$. **(3 p)**

4. The quadratic form Q is given by $Q\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = x^2 - xy + y^2$.

(a) Find the symmetric matrix associated with Q. **(1 p)**

(b) Let $\mathcal{B} = \left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\right\}$. Find the matrix associated with Q in the basis $\mathcal{B}$. **(3 p)**

(c) Determine the type of Q: positivt/negative (semi)definite or indefinite? **(2 p)**

Part C

5. Let $L\colon \mathbb{R}^2 \to \mathbb{R}^3$ an arbitrary but unspecified map.

(a) Why is the dimension of the image $\mathrm{Im}(T)$ of T at most 2? **(1 p)**

(b) Let $\vec{b}$ be a vector in $\mathbb{R}^3$ which lies outside of the range $\mathrm{Im}(T)$. Explain how one can find the vectors $\vec{x}$ minimizing $\|L(\vec{x}) - \vec{b}\|$. **(2 p)**

(c) Apply b) to find the smallest value of $\|L(\vec{x}) - \vec{b}\|$, where $L(\vec{x}) = (x_1, -x_2, x_1 + x_2)$, and $\vec{b} = (1,2,3)$. **(3 p)**

6. Let A be a symmetric $n \times n$ matrix.

(a) Prove that the columns of A are orthonormal if and only if A solves the matrix equation $A^2 = I$. (Here I denotes the identity matrix.) **(2 p)**

(b) Show that if $A^{1246} = I$ then $A^2 = I$. **(4 p)**