Department of Mathematics
SF1626 Calculus in Several Variable

SF1626 Calculus in Several Variable
Solutions to the exam 2017-03-15

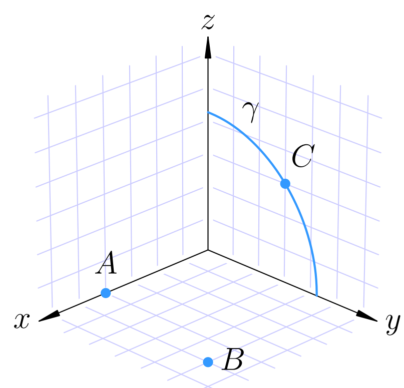
DEL A

1. In the orthogonal coordinate system below each box is one unit long.

(a) Determine the spherical coordinates of the points A, B och C. **(2 p)**

(b) Determine a parametrization in the x,y,z-coordinates of the quarter circular arc γ in the yz-plane with center at the origin. **(1 p)**

(c) Determine a tangent vector to the curve γ at the point C. **(1 p)**



Solution. Spherical coordinates are given by (R, ϕ, θ), where R is the length of the vector, ϕ is the angle towards the z-axis, and θ is the angle in the xy-plane of the projection (longitude).

(a) A: $(R, \phi, \theta) = (4, \pi/2, 0)$.
B: $(R, \phi, \theta) = (\sqrt{50}, \pi/2, \pi/4)$
C: $(R, \phi, \theta) = (\sqrt{18}, \pi/4, \pi/2)$

(b) The circle has radius $\sqrt{18}$ and we use polar coordinates in the yz-plane in order to describe it. Since $x = 0$ on the circle, the parametrization becomes $(x, y, z) = (0, \sqrt{18}\cos t, \sqrt{18}\sin t)$, where $0 \leq t \leq \pi/2$.

(c) Differentiation gives $\big(\dot{x}(\pi/4), \dot{y}(\pi/4), \dot{z}(\pi/4)\big) = \big(0, -\sqrt{18}/\sqrt{2}, \sqrt{18}/\sqrt{2}\big) = (0, -3, 3)$.

Answer.

(a) A: $(R, \phi, \theta) = (4, \pi/2, 0)$. B: $(R, \phi, \theta) = (\sqrt{50}, \pi/2, \pi/4)$ and C: $(R, \phi, \theta) = (\sqrt{18}, \pi/4, \pi/2)$.

(b) $(x, y, z) = (0, \sqrt{18}\cos t, \sqrt{18}\sin t)$, where $0 \leq t \leq \pi/2$.

(c) $(0, -3, 3)$ is a tangent vector to γ at C.

2. A small ball is placed on top of the function surface $z = \dfrac{1}{3}x^2 + \dfrac{1}{4}y^2$ at the point $(3, 2, 4)$ and then starts rolling because of the gravity that is acting in the direction of the negative z-axis.
   (a) In which direction does the ball start rolling if we neglect the z-direction? **(3 p)**
   (b) How steep is it at the point where the ball is placed? **(1 p)**

Solution.

(a) Let $f(x, y) = \frac{1}{3}x^2 + \frac{1}{4}y^2$ and compute the gradient: $\nabla f = (\frac{2}{3}x, \frac{1}{2}y)$. In the point $(x, y) = (3, 2)$ the gradient is $\nabla f(3, 2) = (2, 1)$. The gradient points in the direction where the function is increasing the most, so in the xy-plane the ball will roll in the opposite direction, i.e., $-\nabla f(3, 2) = (-2, -1)$. A normalized directional vector in the xy-plane becomes $\frac{1}{\sqrt{5}}(-2, -1)$.

(b) The directional derivative in the direction of the gradient tells us the maximal slope. In this case the lenght of the the gradient is $\sqrt{2^2 + 1^2} = \sqrt{5}$.

Answer.

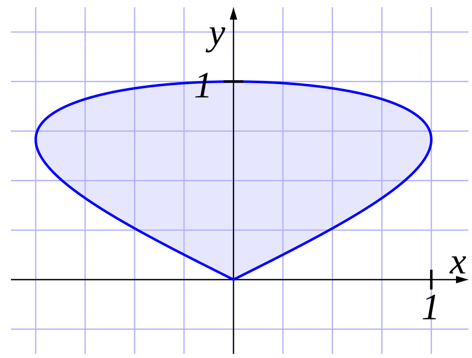
(a) The direction in the xy-plane is $\frac{1}{\sqrt{5}}(-2, -1)$.

(b) The slope is $\sqrt{5}$.

3. The area of a region that is enclosed by a closed, simple curve $\mathcal{C}$ can according to Green's Theorem be calculated by means of a line integral, $\int_{\mathcal{C}} x\,dy$ or $\int_{\mathcal{C}} y\,dx$.

(a) Which orientation should the curve have in order for the first integral to give the area with a positive sign? (Don't forget to motivate your answer.) **(1 p)**

(b) Use one of the two integrals in order to compute the area inside the curve parametrized by $\mathbf{r}(t) = (\sin 2t, \sin t)$, where $0 \le t \le \pi$. **(3 p)**



Solution.

(a) According to Green's Theorem,

$$\int_C P dx + Q dy = \iint_D (Q'_x - P'_y) dx dy.$$

The integral $\int_C x\,dy$ equals the area, under condition that C is positively oriented. This is because $P = 0$ and $Q = x$ gives $Q'_x - P'_y = 1 - 0 = 1$ and the integral of the constant function 1 over the region D enclosed by the curve C gives the area of the region.

(b) The curve is positively oriented so we can use the first integral in order to compute the area. Then we get

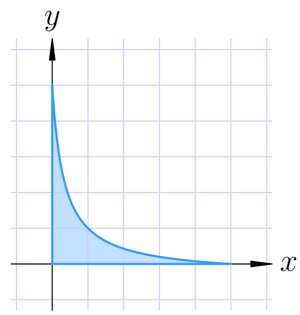
$$\int_C x\,dy = \int_0^\pi \sin(2t)\cos t dt = 2\int_0^\pi \sin t \cos^2 t dt = 2\left[-\frac{\cos^3 t}{3}\right]_0^\pi = \frac{4}{3}$$

which gives the area of the region enclosed by the curve.

Answer. The area is $4/3$.

DEL B

4. Determine the maximum and the minimum value of $f(x, y) = x - xy + y$ in the region D defined by $x \geq 0$, $y \geq 0$, $(x+1)(y+1) \leq 16$.



The region D

(4 p)

Solution. The function f is continuously differentiable and the region D is compact. Hence the maximal and minimal value are attained in one of the following:

(a) interior critical points,
(b) maxima or minima along the boundary,
(c) the vertices.

We get that

(a) The gradient is $\nabla f(x, y) = (1-y, 1-x)$ which equals the null vector precisely when $(x, y) = (1, 1)$. Hence $(1, 1)$ is the only critical point and there we have $f(1, 1) = 1$.

(b) The boundary has three components. We start by investigating the curved part given by the equation $(x + 1)(y + 1) = 16$ by means of Lagrange's method. We use the function

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) = x - xy + y + \lambda((x + 1)(y + 1) - 16)$$

where g expresses the constraint. The critical point condition on the boundary curve is given by the condition that the gradient of F is zero with respect to all three variables:

$$\nabla F(x, y, \lambda) = (1 - y + \lambda(y + 1), 1 - x + \lambda(x + 1), (x + 1)(y + 1) - 16)$$

and we get the system of equations

$$\begin{cases} 1 - y + \lambda(y+1) &= 0, \\ 1 - x + \lambda(x+1) &= 0, \\ (x+1)(y+1) - 16 &= 0. \end{cases}$$

Since $x + 1 \neq 0$ and $y + 1 \neq 0$ we get

$$\lambda = \frac{y-1}{y+1} = \frac{x-1}{x+1}$$

which gives

$$(x + 1)(y - 1) = (x - 1)(y + 1) \iff xy + y - x - 1 = xy + x - y - 1 \iff x = y$$

When substituting this in the last equation we get $(x+1)^2 = 16$, which gives $x = -1 \pm 4$. Since $x \geq 0$ we only get the solution $x = y = 3$ in the given region and the value of the function there is $f(3,3) = -3$.
Alternatively, we can choose a parametrization of the boundary curve as $x = 4e^t - 1$ and $y = 4e^{-t} - 1$ to get the one variable function

$$\begin{aligned} h(t) &= f(4e^t - 1, 4e^{-t} - 1) = 4e^t - 1 - (4e^t - 1)(4e^{-t} - 1) + 4e^{-t} - 1 \\ &= 4e^t - 1 - 16 + 4e^t + 4e^{-t} - 1 + 4e^{-t} - 1 = 8(e^t + e^{-t}) - 19. \end{aligned}$$

The critical points of h is given by the zeroes of $h'(t) = 8(e^t - e^{-t})$. This is zero only at $t = 0$. Hence we are lead to $x = 4 - 1 = 3$ and $y = 4 - 1 = 3$ as before. This gives the minimum along the boundary curve and the maximum is attained at the end points.
It remains to consider the boundary points along the axes. Along the x-axis we get $f(x, 0) = 0$ and the condition that $0 \leq x \leq 15$. Hence the maximum is 15 and the minimum 0. In the same way for the y-axis we get $f(0, y) = y$ where $0 \leq y \leq 15$ with maximum 15 and minimum 0.

(c) The vertices are $(0,0)$, $(15,0)$ and $(0,15)$ where the values of the function are 0, 15 and 15, respectively.

Our candidates for maxima and minima are: the interior critical point $(1,1)$, the point $(3,3)$ along the curved boundary curve, and the points along the linear boundary segments where the extreme values are attained at the end points $(0,0)$, $(15,0)$, and $(0,15)$. We compare the values at these points and we find that $f(1,1) = 1$, $f(3,3) = -3$, $f(0,0) = 0$, $f(15,0) = 15$ och $f(0,15) = 15$. Hence the minimum value is -3 and the maximum valus is 15.

Answer. The minimum values is -3 and the maximal values is 15.

5. Use the change of variables $u = x + y$, $v = y - 2x$ in order to compute the integral

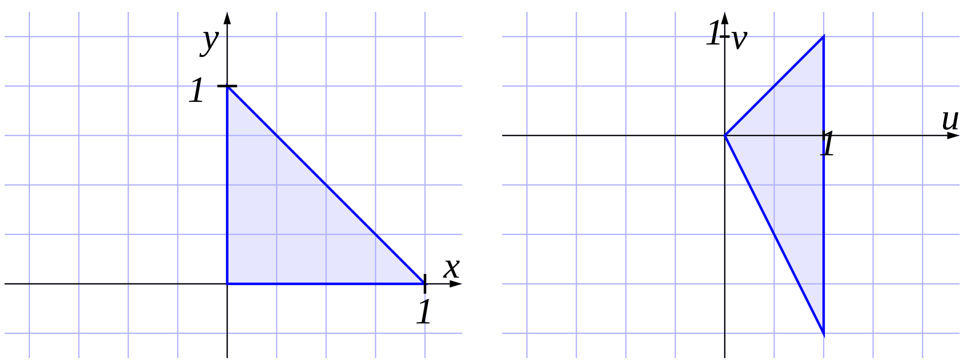
$$\int_0^1 \int_0^{1-x} (y-2x)^2 \sqrt{x+y}\, dy\, dx$$

(4 p)

Solution. The Jacobian in the change of variables is given by

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = 1 \cdot 1 - 1 \cdot (-2) = 3$$

so we get that $dxdy = \frac{1}{3} dudv$. The range of integration gives a triangle given by the conditions $x \geq 0$, $y \geq 0$, and $x + y \leq 1$. The vertices of this triangle are $(0,0)$, $(0,1)$ and $(1,0)$. In order to see how the range of integration changes we can see where the vertices are mapped to since the transformation is linear. We get that $(x,y) = (0,0)$ gives $(u,v) = (0,0)$, $(x,y) = (0,1)$ gives $(u,v) = (0+1, 1-2\cdot 0) = (1,1)$ and $(x,y) = (1,0)$ gives $(u,v) = (1+0, 0-2\cdot 1) = (1,-2)$. This triangle is given by the inequalities $0 \leq u \leq 1$ and $-2u \leq v \leq u$.



FIGUR 1. The two regions

Hence the integral Integralen becomes

$$\int_0^1 \int_{-2u}^{u} v^2 \sqrt{u}\, \frac{dv\, du}{3} = \int_0^1 \left[\frac{v^3}{9}\right]_{-2u}^{u} \sqrt{u}\, du = \int_0^1 u^3 \sqrt{u}\, du = \left[\frac{u^{9/2}}{9/2}\right]_0^1 = \frac{2}{9}.$$

Answer. The value of the integral is $2/9$.

6. Let $\mathbf{F}$ be the vector field in 3-space given by

$$\mathbf{F}(x,y,z) = \left(e^{-y^2-z^2}, e^{-x^2-z^2}, e^{-x^2-y^2}\right).$$

Let $\mathcal{S}$ be the tilted cone without bottom surface that consists of all straight line segments with one end point at $(1,0,3)$ and the other end point at the circle $x^2+y^2=4$ in the xy-plane $z=0$. Use the Divergence Theorem in order to compute the flux of $\mathbf{F}$ up through the surface $\mathcal{S}$ **(4 p)**

Solution. Since

$$\mathbf{F}(x,y,z) = (e^{-(y^2+z^2)}, e^{-(x^2+z^2)}, e^{-(x^2+y^2)})$$

we get that

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}e^{-(y^2+z^2)} + \frac{\partial}{\partial y}e^{-(x^2+z^2)} + \frac{\partial}{\partial z}e^{-(x^2+y^2)} = 0+0+0 = 0.$$

Since the vector field $\mathbf{F}$ is defined over the whole 3-space we can use the Divergenve Theorem in order to move the surface $\mathcal{S}$ to another surface $\mathcal{S}'$ with the same boundary curve without changing the value of the flux of the vector field through the surface. This is because the flux out after closing up the surface becomes zero, and we can conclude that the flux out through the given surface has to equal the flux in through the other surface.

We replace $\mathcal{S}$ by the surface $\mathcal{S}'$ which is given by the circular disc D with radius 2 around the origin in the xy-plane with a normal vector in the positive z-direction, $\hat{\mathbf{N}}' = (0,0,1)$. The flux is now given by

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS &= \iint_{\mathcal{S}'} \mathbf{F}\bullet\hat{\mathbf{N}}'\,dS \\
&= \iint_D (e^{-y^2}, e^{-x^2}, e^{-(x^2+y^2)})\bullet(0,0,1)\,dA \\
&= \iint_D e^{-(x^2+y^2)}\,dA \\
&= \int_0^{2\pi}\left(\int_0^2 e^{-r^2} r\,dr\right)d\theta \\
&= \int_0^{2\pi}\left[-\frac{1}{2}e^{-r^2}\right]_0^2 d\theta \\
&= 2\pi\left(-\frac{1}{2}e^{-4}+\frac{1}{2}\right) \\
&= \pi\left(1-e^{-4}\right).
\end{aligned}$$

Answer. The flux is $\pi(1-e^{-4})$.

DEL C

7. Let $\mathcal{S}$ be the oriented surface in $\mathbb{R}^3$ given by $\mathbf{r}(s,t) = (s, t, st)$ where $s^2 + t^2 \leq 1$ with a normal vector with positive z-component. Let $\mathcal{C}$ be the oriented boundary curve to $\mathcal{S}$ and let the vector field $\mathbf{F}$ be given by
$$\mathbf{F}(x,y,z) = (y,\, xy,\, z^2).$$
Stoke's Theorem relates the flux of the curl of a vector field through a surface with the line integral of the field along the boundary curve. Formulate Stoke's Theorem and use it in order to compute the line integral
$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}} y\,dx + xy\,dy + z^2\,dz.$$
(4 p)

Solution. Stoke's Theorem tells us that
$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \cdot \mathbf{n} dS$$
if $\mathbf{F}$ is a continuously differentiable vector field and $\mathcal{C}$ is the oriented boundary curve of a bounded smooth oriented surface $\mathcal{S}$. The computation of the curl of the given vector field gives that
$$\mathbf{curl}\ \mathbf{F}(x,y,z) = \left(\frac{\partial}{\partial y} z^2 - \frac{\partial}{\partial z} xy, \frac{\partial}{\partial z} y - \frac{\partial}{\partial x} z^2, \frac{\partial}{\partial x} xy - \frac{\partial}{\partial y} y\right) = (0, 0, y-1).$$
Hence we must continue by computing the normal times the area element and integrate according to Stoke's Theorem. We parametrize the surfaca $\mathcal{S}$ by means of polar coordinates, i.e., $x = r\cos\theta$, $y = r\sin\theta$, and $z = xy = r^2\cos\theta\sin\theta$, where $0 \leq r \leq 1$ and $0 \leq \theta < 2\pi$. The surface element with normal direction is
$$\mathbf{n} dS = \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\, dr d\theta = (-r^2\sin\theta, r^2\cos\theta, r) dr d\theta,$$
so we get
$$\iint_{\mathcal{S}} \mathbf{curl}\ \mathbf{F} \cdot \mathbf{n} dS = \int_0^1 \int_0^{2\pi} (r\sin\theta - 1) r\, d\theta dr = -\int_0^1 \int_0^{2\pi} r\, d\theta dr = -\pi,$$
since $\sin\theta$ has mean valus zero and the last integral gives the area of the unit disc.

We can also use the given parametrization of the surface and then compute the flux as the integral of the triple product
$$\mathbf{curl}\,\mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \det \begin{bmatrix} 0 & 0 & t-1 \\ 1 & 0 & t \\ 0 & 1 & s \end{bmatrix} = t - 1.$$
This needs to be integrated over the unit disc where t has mean value 0 and the value of the integral becomes $-\pi$.

Answer. The valus of the line integral is $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}} y\,dx + xy\,dy + z^2\,dz = -\pi.$

8. Let $\mathcal{S}$ be the set of solutions to the equation

$$x^2 + y^2 = z \cos z.$$

(a) Explain how we can be certain that there is a function$f(x, y)$ such that $\mathcal{S}$ in a neighbourhood of the point $(x, y, z) = (0, 0, 0)$ coincides with the graph $z = f(x, y)$. **(2 p)**

(b) Show that $(x, y) = (0, 0)$ is a critical point of this function. f. **(1 p)**

(c) Determine whether this critical point is a local minimum, local maximum or neither. **(1 p)**

Solution.

(a) Let $F(x, y, z) = x^2 + y^2 - z \cos z$. We see that $F(x, y, z) = 0$ for all $x, y, z \in \mathcal{S}$ and $F(0, 0, 0) = 0$, so $(0, 0, 0) \in \mathcal{S}$. We get that $\nabla F = (2x, 2y, z \sin z - \cos z)$. In order to use the Implicit Functions Theorem we need the partial derivatives to be continuous in a neighborhood of $(x, y, z) = (0, 0, 0)$ and $\frac{\partial F}{\partial z} \neq 0$ at this point. We see that these conditions are satisfied and we conclude that the surface $\mathcal{S}$ is given as the graph of a function $z = f(x, y)$ near the point $(0, 0, 0)$.

(b) Write $z = z(x, y)$. Implicit differentiation with respect to x and y gives

$$2x = \cos z \frac{\partial z}{\partial x} - z \sin z \frac{\partial z}{\partial x} \quad \text{and} \quad 2y = \cos z \frac{\partial z}{\partial y} - z \sin z \frac{\partial z}{\partial x}$$

Substitution of $(x, y, z) = (0, 0, 0)$ gives that $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$ i.e., we have a critical point.

(c) Since the left hand side cannot be negative and the right hand side is negative for $-\pi/2 < z < 0$ we have that $f(x, y) \geq 0 = f(0, 0)$ in a neighborhood of the origin. Hence the origin is a local minimum.
We can also see this by means of the Hessian. We need to differentiate once more to get $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$ och $\frac{\partial^2 z}{\partial x \partial y}$.

$$2 = -\sin z \left(\frac{\partial z}{\partial x}\right)^2 + \cos z \frac{\partial^2 z}{\partial x^2} - z \cos z \left(\frac{\partial z}{\partial x}\right)^2 - \sin z \left(z \frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)^2\right)$$

$$2 = -\sin z \left(\frac{\partial z}{\partial y}\right)^2 + \cos z \frac{\partial^2 z}{\partial y^2} - z \cos z \left(\frac{\partial z}{\partial y}\right)^2 - \sin z \left(z \frac{\partial^2 z}{\partial y^2} + \left(\frac{\partial z}{\partial y}\right)^2\right)$$

$$0 = -\sin z \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + \cos z \frac{\partial^2 z}{\partial x \partial y} - z \cos z \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} - \sin z \left(z \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}\right)$$

Substitution of $(x, y, z) = (0, 0, 0)$ gives that $\frac{\partial^2 z}{\partial x^2} = 2$, $\frac{\partial^2 z}{\partial y^2} = 2$ and $\frac{\partial^2 z}{\partial x \partial y} = 0$. We get that the Hessian matrix is positive definite and we have a local minimum.

Answer.

(c) The origin is a local minimum.

9. For a given curve $\mathcal{C}$ in the plane $\mathbb{R}^2$ we can define the average distance between two points on $\mathcal{C}$ as
$$\bar{d}(\mathcal{C}) = \frac{1}{L^2}\int_{\mathcal{C}}\int_{\mathcal{C}} |\mathbf{r}(s) - \mathbf{r}(t)|\, dsdt,$$
where L is the length of $\mathcal{C}$ and $\mathbf{r}(t)$ is the arc-length parametrization of $\mathcal{C}$.
   (a) Compute $\bar{d}(\mathcal{C})$ where $\mathcal{C}$ is the line segment from $(0,0)$ to $(1,1)$. **(2 p)**
   (b) Compute $\bar{d}(\mathcal{C})$ where $\mathcal{C} = S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, the unit circle in the plane. **(2 p)**

Solution.

(a) We use the acr length parametrization of the line segment $y = x$ according to $x = \frac{s}{\sqrt{2}}$ i.e., $\mathbf{r}(s) = \frac{1}{\sqrt{2}}(s,s)$ which gives that $|\mathbf{r}(s) - \mathbf{r}(t)| = |s-t|$. The length is $L = \sqrt{2}$ and we get
$$\bar{d}(\mathcal{C}) = \frac{1}{2}\int_0^{\sqrt{2}}\int_0^{\sqrt{2}} |s-t|\, dsdt = \frac{1}{2}\int_0^{\sqrt{2}}\left(\int_0^t (t-s)ds + \int_t^{\sqrt{2}}(s-t)ds\right) dt$$
$$= \frac{1}{2}\int_0^{\sqrt{2}}\left(t^2 - \sqrt{2}t + 1\right) dt = \frac{\sqrt{2}}{3}.$$

(b) The arc length parametrization of the circle is $x = \cos t, y = \sin t$ and we get that
$$|\mathbf{r}(s) - \mathbf{r}(t)| = \sqrt{(\cos s - \cos t)^2 + (\sin s - \sin t)^2} = \sqrt{2 - \cos(s-t)}$$
$$= \sqrt{2 - 2\left((1 - 2\sin^2\frac{(s-t)}{2}\right)} = 2\left|\sin\frac{(s-t)}{2}\right|$$
We see that $|\sin\frac{(s-t)}{2}|$ is a periodic function and that we integrate over one period. Hence the translation by t doesn't change the value of the integral. The length of the circle is 2π. Hence we get
$$\bar{d}(\mathcal{C}) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} 2\left|\sin\frac{(s-t)}{2}\right|\, dsdt = \frac{1}{4\pi^2}\cdot 2\pi\int_0^{2\pi} 2\left|\sin\frac{s}{2}\right|\, ds$$
$$= \frac{1}{2\pi}\int_0^{2\pi} 2\sin\frac{s}{2}\, ds = \frac{1}{\pi}\left[-2\cos\frac{s}{2}\right]_0^{2\pi} = \frac{1}{\pi}\left(-2(-1) + (2)\right) = \frac{4}{\pi}$$

Answer.

(a) $\bar{d}(\mathcal{C}) = \sqrt{2}/3$ for the line segment from $(0,0)$ to $(1,1)$.
(b) $\bar{d}(\mathcal{C}) = 4/\pi$ for the unit circle.