Department of Mathematics
SF1626 Calculus in Several Variable

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Solutions to the exam 2017-06-05

DEL A

1. The height of a hill is given by $z = 60 - 0{,}02x^2 - 0{,}01y^2$ where the unit is meter on all three coodrinate axes.
   (a) In which direction in the xy-plane should we move in order to descend fastest possible if we are at the point $(50, 100, -90)$? **(2 p)**
   (b) What is the rate of change in height at the point $(50, 100, -90)$ if we move in the direction according to part (a) with a speed of 1 km/h seen from above? **(2 p)**

Solution.

(a) The function $z(x, y)$ grows fastest in the direction of its gradient

$$(-0{,}02 \cdot 2x, -0{,}01 \cdot 2y) = \frac{1}{100}(-4x, -2y).$$

Hence, the fastest descent is given by the reverse direction

$$-\frac{1}{100}(-4 \cdot 50, -2 \cdot 100) = (2, 2).$$

(b) The directional derivative in the direction of the negative gradient is

$$-|(2, 2)| = -2\sqrt{2}.$$

Moving with speed 1000 m/h in xy-direction gives us the following rate of change in z-direction

$$2\sqrt{2} \cdot 1000\,\text{m/h}.$$

Answer.

(a) $(1/\sqrt{2}, 1/\sqrt{2})$.
(b) $2\sqrt{2}$ km/h, approximately $2{,}8$ km/h.

2. Prove the formula $V = \dfrac{4\pi a^3}{3}$ for the volume of a spherical ball with radius a by introducing spherical coordinates in the triple integral

$$V = \iiint_K dV,$$

where the ball K is given by $x^2 + y^2 + z^2 \leq a^2$. **(4 p)**

Solution. We know that $\iiint_K dV$ gives the volume of the ball K. In spherical coordinates the ball with radius a and center at the origin is given by $0 \leq r \leq a$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$. Hence the integral in spherical coordinates is

$$\begin{aligned}\iiint_K dV &= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^a r^2 \, dr \\ &= [\theta]_0^{2\pi} \, [-\cos\phi]_0^{\pi} \left[\frac{r^3}{3}\right]_0^a \\ &= 2\pi \cdot (-(-1) - (-1)) \cdot \frac{a^3}{3} \\ &= \frac{4\pi a^3}{3},\end{aligned}$$

which proves the folrumla.

3. Determine which of the fields
$$\mathbf{F}(x,y,z) = \big(x(z-1), -yz, z-x^2\big)$$
and
$$\mathbf{G}(x,y,z) = \big(y^2+(2x-y)z, x(2y-z), x(x-y)\big)$$
that is conservative and determine a potential for that field. **(4 p)**

Solution. Observe that irrotation fields defined in the entire $\mathbb{R}^3$ are always conservative fields, otherwise they may not necessarily be conservative.

Computations give
$$\begin{aligned}\mathbf{rot}\,\mathbf{F} &= \left(\frac{\partial}{\partial y}(z-x^2) - \frac{\partial}{\partial z}(-yz), \frac{\partial}{\partial z}x(z-1) - \frac{\partial}{\partial x}(z-x^2), \frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}x(z-1)\right)\\ &= (0-(-y), x-(-2x), 0-0) = (y,x,0) \neq (0,0,0)\end{aligned}$$
$$\begin{aligned}\mathbf{rot}\,\mathbf{G}(x,y,z) = \Bigg(&\frac{\partial}{\partial y}x(x-y) - \frac{\partial}{\partial z}x(2y-z), \frac{\partial}{\partial z}(y^2+(2x-y)z) - \frac{\partial}{\partial x}x(x-y),\\ &\frac{\partial}{\partial x}x(2y-z) - \frac{\partial}{\partial y}(y^2+(2x-y)z)\Bigg)\\ = (&-x-(-x), 2x-y-(2x-y), 2y-z-(2y-z)) = (0,0,0).\end{aligned}$$
Hence $\mathbf{G}$ is irrotational, but not $\mathbf{F}$. To determine the potential function $\mathbf{G}$ we integrate first in x-direction to obtain
$$\Phi(x,y,z) = xy^2 + x^2z - xyz + H(y,z).$$
Differentiating $\Phi(x,y,z)$ w.r.t. y and z we arrive at
$$\frac{\partial\Phi}{\partial y} = 2xy - xz + \frac{\partial H}{\partial y} \quad \text{och} \quad \frac{\partial\Phi}{\partial z} = x^2 - xy + \frac{\partial H}{\partial z}$$
and sine $2xy - xz = x(2y-z)$ and $x^2 - xy = x(x-y)$ can we choose (it is up to us which one we choose) $H(y,z) = 0$. Therefore $\Phi(x,y,z) = xy^2 + x^2z - xyz$ is a potential function to $\mathbf{G}$.

Answer. The vector field $\mathbf{G}$ is conservative with potential function $\Phi(x,y,z) = xy^2 + x^2z - xyz$. The vector filed $\mathbf{F}$ is not conservative.

DEL B

4. The elliptic cylinder $9x^2 + 25y^2 = 225$ and the plane $4y + 3z = 0$ intersect in a curve $\mathcal{C}$.
   (a) Give a parametrization of the curve $\mathcal{C}$. **(2 p)**
   (b) Compute the length of the curve $\mathcal{C}$. **(2 p)**

Solution.

(a) The ellipsen E in the plane $z = 0$ is given by $9x^2 + 25y^2 = 225$, i.e.,

$$\frac{1}{5^2}x^2 + \frac{1}{3^2}y^2 = 1.$$

Parametrising this we have $(5\cos(t), 3\sin(t))$, where $0 \leq t \leq 2\pi$. The curve we look for is above the ellipse E, for which z is given by $4y + 3z = 0$, i.e. $z = -\frac{4}{3}y$. Hence the curve $\mathcal{C}$ is given by

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = (5\cos t, 3\sin t, -4\sin t)$$

för $0 \leq t \leq 2\pi$.

(b) The length of the curve is given by

$$\begin{aligned}
\int_{\mathcal{C}} ds &= \int_0^{2\pi} |\mathbf{r}'(t)|\, dt \\
&= \int_0^{2\pi} \sqrt{(-5\sin t)^2 + (3\cos t)^2 + (-4\cos t)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{25\sin^2 t + 9\cos^2 t + 16\cos^2 t}\, dt \\
&= \int_0^{2\pi} \sqrt{25}\, dt = 2\pi \cdot 5 = 10\pi.
\end{aligned}$$

(Observe that the curve is a circle with radius 5. If we had chosen intersection with a different plane, the answer could well have been an ellipse. Such an integral would not have been easy to compute.)

Answer.

(a) A parametrisation is given by $\mathbf{r}(t) = (x(t), y(t), z(t)) = (5\cos t, 3\sin t, -4\sin t)$.
(b) The lenght of the curve is 10π length unit.

5. Let $f(x,y) = x^2 + 2xy - 5y^2$ and $g(x,y) = x^2 - 2xy + 2y^2$.
   (a) Determine the maximum and minimum values for $f(x,y)$ given that $g(x,y) = 10$ if such values exist. **(2 p)**
   (b) Determine the maximum and minimum values for $g(x,y)$ given that $f(x,y) = 10$ if such values exist. **(2 p)**

Solution. By Lagranges condition $\nabla f(x,y) = (2x+2y, 2x-10y)$ and $\nabla g(x,y) = (2x-2y, -2x+4y)$, should be parallell. This can be written as

$$\begin{cases} 2x+2y &= \lambda(2x-2y) \\ 2x-10y &= \lambda(-2x+4y) \end{cases} \quad \text{or} \quad \begin{cases} 2x-2y &= \lambda(2x+2y) \\ -2x+4y &= \lambda(2x-10y) \end{cases}$$

This in turn gives $(2x+2y)(-2x+4y) = (2x-2y)(2x-10y)$ which can be simplified to $2x^2 - 7xy + 3y^2 = 0$. Solving these we arrive at

$$x = \frac{7y}{4} \pm y\sqrt{\frac{49}{16} - \frac{3}{2}} = \frac{7y}{4} \pm y\sqrt{\frac{25}{16}} = \frac{7y}{4} \pm \frac{5y}{4}$$

i.e., $x = 3y$ or $x = y/2$.

(a) The solutions set to $g(x,y) = 10$ gives an ellipse, since we can rewrite it as $(x-y)^2 + y^2 = 10$. Frome here we conclude that both maximum and minimum for $f(x,y)$ are achieved on this compact set. Since there are no singular points Lagrange condition must be fulfilled. Inserting $x = 3y$ in the equation we obtain $9y^2 - 6y^2 + 2y^2 = 10$, i.e, $5y^2 = 10$. Therefore the solution is give by $y = \pm\sqrt{2}$ and we have $f(3y,y) = 9y^2 + 6y^2 - 5y^2 = 10y^2 = 20$. When we insert $x = y/2$, i.e., $y = 2x$ in the equation we obtain $x^2 - 4x^2 + 8x^2 = 10$, i.e., $5x^2 = 10$ and $x = \pm\sqrt{2}$ and we have $f(x,2x) = x^2 + 4x^2 - 20x^2 = -15x^2 = -30$. Det largest value is $f(3\sqrt{2}, \sqrt{2}) = f(-3\sqrt{2}, -\sqrt{2}) = 20$ and the smallest one $f(\sqrt{2}, 2\sqrt{2}) = f(-\sqrt{2}, -2\sqrt{2}) = -30$.

(b) The solution set to $f(x,y) = 10$ is an unbounded curve. We can rewrite this as $(x+y)^2 - 6y^2 = 10$, which gives $x = -y \pm \sqrt{10 + 6y^2}$. Therefore the function $g(x,y)$ can obtain any large value. However, $g(x,y)$ is bounded from below, since $g(x,y) = (x-y)^2 + y^2 \geq 0$. The smallest value for the function must be achieved on a compact set, and Lagrange condition must be fulfilled, since there are no singular points.
Setting $x = 3y$ in the equation we obtain $9y^2 + 6y^2 - 5y^2 = 10$, i.e., $10y^2 = 10$. From here we conclude that solutions are given by $y = \pm 1$, which implies $g(3y,y) = 9y^2 - 6y^2 + 2y^2 = 5y^2 = 5$. When we set in $x = y/2$, i.e., $y = 2x$ in the equation we obtain $x^2 + 4x^2 - 20x^2 = 10$, i.e., $-15x^2 = 10$ which has no solution.
The function has no largest value (it is unbounded from above) and its smallest value is given by $f(3,1) = f(-3,-1) = 5$.

Answer.
(a) Maximum is 20 and minimum -30.
(b) Maximum is not achieved and minimum is 5.

6. A snowball has the form of a spherical ball with center in the origin and with radius a. It is lit by the sun which is far away in the positive y-direction.

   Determine the incoming effect
$$-\iint_S (0,-I,0)\cdot \hat{\boldsymbol{n}}\, dS,$$
   where S is the part of the surface of the ball that satisfies $y>0$ and hence is lit by the sun, I is the intesity of the sun, $\hat{\boldsymbol{n}}$ is the normal direction of the surface and dS is the area element. **(4 p)**

Solution. The surface are of upper hemisphere can be described in spherical coordinates as
$$r=a,\quad \phi\colon 0\to\pi,\quad \theta\colon 0\to\pi.$$
In particular
$$\begin{aligned}\hat{\boldsymbol{n}}\, dS &= r\sin\phi\, \boldsymbol{r}\, d\phi\, d\theta\\ &= a\sin\phi\,(a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi)\, d\phi\, d\theta.\end{aligned}$$
The incoming effect of the ray is
$$\begin{aligned}-\iint_S (0,-I,0)\cdot\hat{\boldsymbol{n}}\, dS &= \iint_{\substack{\phi\colon 0\to\pi\\ \theta\colon 0\to\pi}} a\sin\phi\cdot a\sin\phi\sin\theta\, I\, d\phi\, d\theta\\ &= a^2 I\int_0^\pi \sin^2\phi\, d\phi\int_0^\pi \sin\theta\, d\theta\\ &= a^2 I\int_0^\pi\left(\frac{1-\cos(2\phi)}{2}\right) d\phi\int_0^\pi\sin\theta\, d\theta\\ &= a^2 I\cdot\left[\frac{\phi}{2}-\frac{\cos(2\phi)}{4}\right]_0^\pi\cdot[-\cos\theta]_0^\pi\\ &= a^2 I\cdot\pi/2\cdot 2\\ &= \pi a^2 I.\end{aligned}$$

One may possibly use the divergence theorem to solve the problem. Since the vector filed is constant and hence divergence free. Add a two-dimensional disk to the problem so that you have a complete closed surface and use divergence theorem. Then you have to subtract the two-dimensional disk, from the problem. The computation on the disk is easier.

Answer. The incoming effect is $\pi a^2 I$.

DEL C

7. A region that lies on one side of a plane section through a sphere is called a *spherical cap*. Determine the area of the spherial cap given by
$$x^2 + y^2 + z^2 = a^2, \quad z \geq h,$$
where a and h are constants with $0 < h < a$. **(4 p)**

Solution. Through parametrisation of the spherical cap we have
$$\mathbf{r}(x, y) = (x, y, \sqrt{a^2 - x^2 - y^2})$$
where $(x, y) \in D$, and the latter is given by $x^2 + y^2 \leq a^2 - h^2$. The area of the cap Y is then

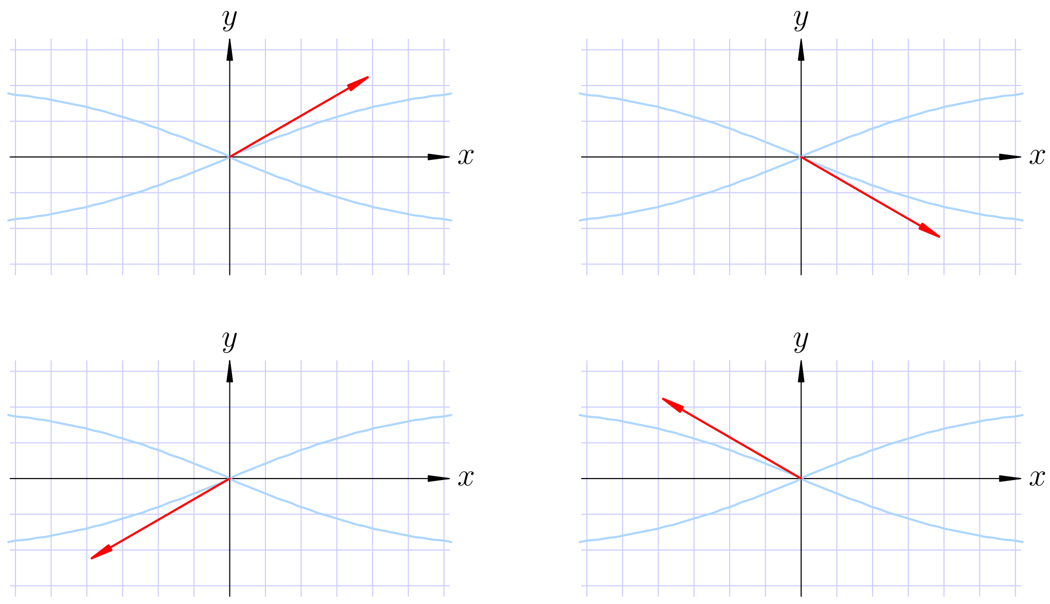
$$\begin{aligned}
\iint_Y dS &= \iint_D \left| \left(\frac{x}{\sqrt{a^2 - x^2 - y^2}}, \frac{y}{\sqrt{a^2 - x^2 - y^2}}, 1 \right) \right| dxdy \\
&= \iint_D \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1}\, dxdy \\
&= \int_0^{2\pi} d\theta \int_0^{\sqrt{a^2 - h^2}} \left(\sqrt{\frac{r^2}{a^2 - r^2} + 1} \right) r\, dr \\
&= 2\pi \int_0^{\sqrt{a^2 - h^2}} \left(\sqrt{\frac{r^2 + a^2 - r^2}{a^2 - r^2}} \right) r\, dr \\
&= 2\pi a \int_0^{\sqrt{a^2 - h^2}} \left(\frac{1}{\sqrt{a^2 - r^2}} \right) r\, dr = \{u = a^2 - r^2, du = -2rdr\} \\
&= 2\pi a \int_{a^2}^{h^2} \frac{1}{\sqrt{u}} \left(-\frac{1}{2} \right) du \\
&= \pi a \int_{h^2}^{a^2} \frac{1}{\sqrt{u}}\, du = \pi a \left[2\sqrt{u} \right]_{h^2}^{a^2} = 2\pi a(a - h).
\end{aligned}$$

We may also use spherical coordinates through $r = a$, where $0 \leq \phi \leq \arccos(h/a)$ and $0 \leq \theta \leq 2\pi$, which gives the area

$$\begin{aligned}
\int_0^{2\pi} \int_0^{\arccos(h/a)} a^2 \sin\phi\, d\phi d\theta &= a^2 \left[\theta\right]_0^{2\pi} \left[-\cos\phi\right]_0^{\arccos(h/a)} \\
&= 2\pi a^2(-h/a - (-1)) = 2\pi a(a - h).
\end{aligned}$$

Answer. The area of the cap is $2\pi a(a - h)$ unit area.

8. At the origin, the curve $\cos(x+y)+\cos(x-y)+4y^2 = 2$ has a branch point and it decomposes into four curve segments meeting there. Determine the directional vectors for these curve segments at the origin.



The curve $\cos(x+y)+\cos(x-y)+4y^2 = 2$ and the direction vectors in question.

(4 p)

Solution. By Taylor's formula $\cos t = 1 - t^2/2 + O(t^3)$, applied to the left hand side of the equation, we have

$$2 - x^2 + 3y^2 + O(r^3) = 2 \quad \Leftrightarrow \quad (\sqrt{3}y - x)(\sqrt{3}+y) + O(r^3) = 0,$$

where $r = |(x,y)| = \sqrt{x^2+y^2}$.

Divide both sides with $r^2 = x^2+y^2$,

$$\frac{\sqrt{3}y - x}{\sqrt{x^2+y^2}} \cdot \frac{\sqrt{3}y + x}{\sqrt{x^2+y^2}} + O(r) = 0.$$

Letting $(x,y) \to (0,0)$, we have

$$\frac{\sqrt{3}y - x}{\sqrt{x^2+y^2}} \cdot \frac{\sqrt{3}y + x}{\sqrt{x^2+y^2}} \to 0.$$

By continuity of the above expressions (in the punctured disk around the origin) and that the curve is connected we obtain

$$① \; \frac{\sqrt{3}y - x}{\sqrt{x^2+y^2}} \to 0 \quad \text{or} \quad ② \; \frac{\sqrt{3}y + x}{\sqrt{x^2+y^2}} \to 0,$$

i.e.,

$$\text{①}\ x = \sqrt{3}y + o(r) \quad \text{eller} \quad \text{②}\ x = -\sqrt{3}y + o(r).$$

In case ① we have

$$\frac{(x,y)}{\sqrt{x^2+y^2}} = \frac{(\sqrt{3}y, y) + o(r)}{\sqrt{y^2+3y^2+o(r^2)}} = \frac{y}{|y|}\,\tfrac{1}{2}(\sqrt{3},1) + \frac{o(r)}{r} \to \begin{cases} \frac{1}{2}(\sqrt{3},1), & \text{om } k \to 0+ \\ -\frac{1}{2}(\sqrt{3},1), & \text{om } k \to 0- \end{cases}$$

In case ② we have

$$\frac{(x,y)}{\sqrt{x^2+y^2}} = \frac{(-\sqrt{3}y, y) + o(r)}{\sqrt{y^2+3y^2+o(r^2)}} = \frac{y}{|y|}\,\tfrac{1}{2}(-\sqrt{3},1) + \frac{o(r)}{r} \to \begin{cases} \frac{1}{2}(-\sqrt{3},1), & \text{om } k \to 0+ \\ \frac{1}{2}(\sqrt{3},-1), & \text{om } k \to 0- \end{cases}$$

The four directions are är alltså $\pm\frac{1}{2}(\sqrt{3},1)$ and $\pm\frac{1}{2}(-\sqrt{3},1)$.

An alternative way is to rewrite the equation as

$$2\cos x\cos y + 4y^2 = 2.$$

The gradient of the left hand side is är $(-2\sin x\cos y, 8y^2 - \cos x\sin y)$ and we can solve y as a function of $x$by implicit funktion theorem. Differentiating the equation twice we obtain

$$2\sin x\cos y - 2y'\cos x\sin y + 8y'y = 0$$

and

$$-2\cos x\cos y + 2y'\sin x\sin y - 2y''\cos x\sin y + 2y'\sin x\sin y$$
$$-2(y')^2\cos x\cos y + 8y''y + 8(y')^2 = 0$$

Uppon approaching the origin the limits of $\sin x$, $\sin y$ and y wil be zero, while the limit of $\cos x$ and $\cos y$ is one. If α is the limit for y' and β for y'' we obtain

$$-2\cdot 1\cdot 1 + 2\alpha\cdot 0\cdot 0 - 2\beta\cdot 1\cdot 0 + 2\beta\cdot 0\cdot 0 - 2\alpha^2\cdot 1\cdot 1 + 8\beta\cdot 0 + 8\alpha^2 = 0$$

which gives $6\alpha^2 = 2$, i.e., $\alpha = \pm 1/\sqrt{3}$. The directions will be as mentioned above.

Answer. The four directions are $\frac{1}{2}(\pm\sqrt{3},\pm 1)$.

9. Let the curve $\mathcal{C}$ be the triangle with vertices at the points $(3,4)$, $(-4,0)$, $(2,-3)$, oriented clockwise. Compute
$$\oint_{\mathcal{C}} \frac{-ydx + xdy}{x^2+y^2}.$$
(4 p)

Solution. The vector field
$$\mathbf{F} = (F_1, F_2) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$
satisfies
$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$
outside the origin. We may apply Greens theorem in a different way, and change the integration path, as follows. Define a new curve $\mathcal{C}'$ (this is the unit circle) parametrized by
$$(x(t), y(t)) = (\cos t, \sin t), \qquad 0 \le t \le 2\pi.$$
The integral we are looking for can now be transformed (through Greens theorem) as follows
$$\begin{aligned}\oint_{\mathcal{C}} \frac{-ydx + xdy}{x^2+y^2} &= \oint_{\mathcal{C}'} \frac{-ydx + xdy}{x^2+y^2} \\ &= \int_0^{2\pi} \frac{-\sin t(-\sin t) + \cos t \cos t}{\cos^2 t + \sin^2 t}\, dt \\ &= \int_0^{2\pi} dt \\ &= 2\pi.\end{aligned}$$

Answer. $\displaystyle\oint_{\mathcal{C}} \frac{-ydx + xdy}{x^2+y^2} = 2\pi.$