MATH.SE

SVERIGES UNIVERSITETS MATEMATIKPORTAL

Preparatory Course in Mathematics 1

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Contents

Welcome to the course

There is now an easy way to be better prepared for your university studies

This course is meant for those of you whose studies at university will include mathematics and who wish to be well prepared before the start of the degree course.

It is designed as a bridging course between secondary school and university. We recommend it to you even if you have done well in mathematics at secondary school. It is eligible for student financial assistance and can be followed entirely on the Internet.

You make your own timetable for your studies, so you can easily adapt them to fit in with any other plans you may have.

Registration and access to forums, support, examination and a personal mentor

The course literature is open and accessible via the Internet. You can register on the course anytime during the year by means of the electronic form on www.math.se and you will be given a user name and password that provides access to all course materials, discussion forums, support, monitoring and tests. You will also be assigned a personal mentor to help you do well in your studies.

How to get the most out of this course:

1. In any given section, start by reading the review of the theory and then work through the examples.
2. Then, work through the practice problems and try to solve them without using a calculator. Check whether you have got the right answer by clicking on the answer button. If you do not have the right answer, you can click on the solution button to see how the problem is done.
3. Go on to answer the questions in the basic test.
4. If you get stuck, see if this particular topic has been discussed in the forum. Pose a question if you have any queries. Your teacher (or a student) will respond to it within a few hours.
5. When you are finished with the exercises and the basic test belonging to a section, you must do a final test to obtain a pass for the section. Here the requirement is answering correctly three questions one after the other in order to move on.

6. If you have answered correctly in both the basic and the final test, you have passed that particular section and may advance to the next part of the course.

P.S. If you feel that you are familiar with the contents of a section you can go directly to the basic and final tests. To pass, you must get all the answers right, but you may redo the test several times if you do not succeed at first attempt. It is your final results that appear in the course statistics.

Supervision and examination

You may at any time take part in online discussions with fellow students, ask questions and receive guidance from teachers. Examination is via the Internet as you work on the course.

Please note that the course has been designed on the assumption that calculators are not used

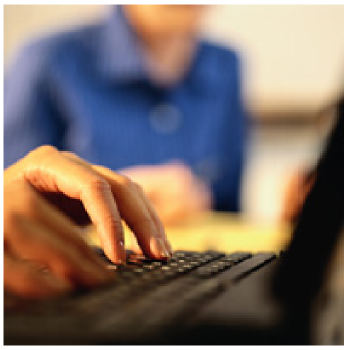
Calculator use in University mathematics varies from course to course; some Departments do not allow it at all, while others permit calculators some of the time. For most tasks, there is in any case no advantage in using a calculator.

How is the course structured?

Up-to-date knowledge increases your chances of success

This is a bridging course between secondary school and university. It goes through some of the skills and knowledge that we think are so important that they ought to be refreshed before your university studies. It is flexible: you study at a pace that suits you.

Here is how it is envisaged that you will work with the course:



- In any given section, start by reading the review and then study the examples.
- Work through the exercises and then answer the questions in the basic test belonging to the section. If you get stuck, see if someone has asked a question about this particular part of the course in the forum, or else ask a question yourself.
- When you are finished with the exercises and the basic test in one section, take the final test of the section to obtain a pass for that section.
- When you have passed all the final tests you will be given an individual assignment that you should solve independently and submit, after which you will work in a group.

Your personal mentor assists you

When you log in with your user name you will be taken to the "Student lounge." Here you will find the email address and telephone number of your personal mentor, whom you can contact if you get stuck on a question or have anything else you need help with.

The mentors are teachers and students at the universities taking part in this course. Our common goal is that everybody who starts on the course manages to complete it and thus has a good grounding for their future university studies. For us, there no such thing as a stupid question; if something is not clear to you, the chances are you are not alone!

Examinations

Examinations are online

The exam consists of two computer-marked tests per section; in addition, at the end of the course you will do both an individual assignment and a group project. The course grade is obtained when all five parts are approved. Your grade will be a straight pass or fail.

The basic and final tests are computer marked

For each section of the course, there is both a basic and a final test. Links to the tests can be found in the "Student Lounge" that you arrive at when you log in with your personal user name. You cannot fail these tests; if you do not pass, you can just take the test again until you do. To pass one of these tests you must get all the questions right.

The final tests consist of three randomly generated questions that are automatically marked by the computer. Here, you are expected to solve a problem on paper and input your answer. You must answer all three questions correctly at the same sitting to pass.

If you give the wrong answer to any question, you can make another attempt. You will now be presented with three new versions of the questions; even if you passed one or more of the previous questions, you start from scratch and have to pass all three questions again. However, keep in mind that it is your final performance that is recorded in the statistics of the course.

The assignments are an essential part of the exam

Assignments make up Part 5 of the course. Via a link on the home page of the course for assignments, you will be able to download your individual assignment when you are halfway into the course. When you have passed three-quarters of all the tests you may submit your solution to the assignments.

In the assignments, you are expected to be able to present an idea or an argument in your own words and not just give an answer or choose an alternative. Your individual assignment need not be perfect; it is only at the next stage — when you are working together with other participants — that the solutions need to be finalised.

Group assignments teach you to discuss mathematics with others

When you submit your solution to your individual assignment, you will be automatically grouped with three other people who recently submitted their individual submissions. The group is automatically assigned its own group forum, where members can communicate with each other; at the group forum there is an icon to click on when the group is ready to submit the joint project.

The group's task is to review all the members' individual submissions and then agree on a common 'best' solution to each of the individual projects.

The group will then make a joint submission and it is these solutions that are reviewed and commented on by the teacher. The teacher communicates with the entire group, and if there is anything the group has missed you have the opportunity to make a new group submission until everything is satisfactory and can be approved. To pass this part of the course you need to participate actively (by asking questions, for example) and to help with work on the joint final submission.

Wait to submit your assignment if you are planning to be away from the course.

It sometimes can take a few days to be assigned to a group, but usually it goes much faster. Once a group is formed it is expected to begin work on the assignment at once.

For example, if you are away and you do not have access to the Internet then this will cause problems for the other group members as they cannot begin work with the group assignment. Therefore, if you know that you will be away from the course, please wait to submit your assignment until you have returned and can be active in the group.

1.1 Different types of numbers

Contents:

- Natural numbers
- Negative numbers
- Order of precedence and parenthesis
- Rational numbers
- Briefly about irrational numbers
- Real numbers

Learning outcomes:

After this section you will have learned to:

- Calculate the value of an expression that contains integers, the four arithmetic operations and parentheses (brackets).
- Know the difference between the natural numbers, integers, rational numbers and irrational numbers.
- Convert fractions to decimals and vice versa.
- Determine which of two fractions is the larger, either by a decimal expansion or by cross multiplication.
- Determine the approximate value of a decimal number or a fraction to a given number of decimal places.

Calculations with numbers

Calculating with numbers requires you to perform a series of operations. These are the four basic operations of arithmetic.

When you add numbers the sum does not depend on the order of the terms.

$$3+4+5=3+5+4=5+4+3=12.$$

In the case of subtraction, of course, the order is important.

$$5-2=3 \quad \text{whereas} \quad 2-5=-3.$$

When we talk about the difference between two numbers we usually mean the difference between the larger and the smaller. Thus, we say the difference between 2 and 5 is 3.

Addition

$$\overset{\text{term}}{3} + \overset{\text{term}}{4} = \overset{\text{sum}}{7}$$

Subtraction

$$\overset{\text{term}}{13} - \overset{\text{term}}{4} = \overset{\text{difference}}{9}$$

Multiplication

$$\overset{\text{factor}}{3} \cdot \overset{\text{factor}}{4} = \overset{\text{product}}{12}$$

Division

$$\frac{\overset{\text{numerator}}{8}}{\underset{\text{denominator}}{4}} = \overset{\text{quotient}}{2}$$

When numbers are multiplied their order is not important.

$$3 \cdot 4 \cdot 5 = 3 \cdot 5 \cdot 4 = 5 \cdot 4 \cdot 3 = 60.$$

With division, the order matters.

$$\frac{6}{3} = 2 \quad \text{whereas} \quad \frac{3}{6} = 0{,}5.$$

Hierarchy of arithmetic operations (priority rules)

If several mathematical operations occur in a mathematical expression it is important to have a standard for the order in which the operations are to be carried out. The following order applies:

- Parentheses (brackets, “innermost brackets” first)
- Multiplication and division (from left to right)
- Addition and subtraction (from left to right)

Example 1

a) $3 - (2 \cdot (3+2) - 5) = 3 - (2 \cdot 5 - 5) = 3 - (10 - 5) = 3 - 5 = -2$

b) $3 - 2 \cdot (3+2) - 5 = 3 - 2 \cdot 5 - 5 = 3 - 10 - 5 = -7 - 5 = -12$

c) $5 + 3 \cdot \left(5 - \frac{-4}{2}\right) - 3 \cdot (2 + (2-4)) = 5 + 3 \cdot (5 - (-2)) - 3 \cdot (2 + (-2))$

$= 5 + 3 \cdot (5+2) - 3 \cdot (2-2) = 5 + 3 \cdot 7 - 3 \cdot 0 = 5 + 21 - 0 = 26$

“Invisible parentheses”

When a division is expressed as a fraction, the numerator and the denominator must be calculated separately before the division is carried out. One can therefore say that there are “invisible parentheses” around the numerator and denominator.

Example 2

a) $\dfrac{7+5}{2} = \dfrac{12}{2} = 6$

b) $\dfrac{6}{1+2} = \dfrac{6}{3} = 2$

c) $\dfrac{12+8}{6+4} = \dfrac{20}{10} = 2$

This is especially important to remember if you are using a calculator.

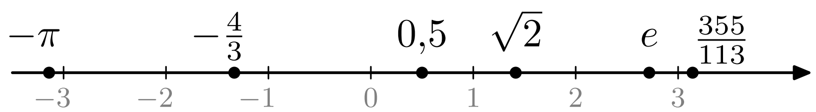
For example, the division

$$\frac{8+4}{2+4}$$

must be written as $(8+4)/(2+4)$ for a calculator so that the correct answer 2 may be obtained. A common mistake is to write $8+4/2+4$, which the calculator interprets as $8+2+4=14$.

Different types of numbers

The numbers we use to describe quantity, size, etc are generically called the real numbers and can be illustrated by a straight line: the real-number axis:



The real numbers “fill” the real-number axis; that is, there are no holes or spaces along it. Each point on the real-number axis can be specified by a decimal. The set of real numbers are all the decimals and is denoted by **R**. The real-number axis also shows the relative magnitude of numbers; a number to the right is always greater than a number to the left. It is standard to classify the real numbers into the following types:

Natural numbers (usually symbolised by the letter ***N****)*

The numbers which are used when we calculate “how many”: 0, 1, 2, 3, 4, . . .

*Integers (**Z**)*

The natural numbers and their negative counterparts: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

*Rational numbers (**Q**)*

All the numbers that can be written as a ratio of whole numbers (fractions). For example:

$$-\frac{3}{4}, \frac{3}{2}, \frac{37}{128}, \quad \text{etc.}$$

Note that integers are classed as rational numbers because

$$-1 = \frac{-1}{1}, \quad 0 = \frac{0}{1}, \quad 1 = \frac{1}{1}, \quad 2 = \frac{2}{1}, \quad \text{etc.}$$

A rational number can be written in various ways. For example:

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \frac{100}{50} = \frac{384}{192} \quad \text{etc.}$$

Example 3

a) Multiplying the numerator and denominator of a rational number by the same factor does not change the value of the number.

$$\frac{1}{3} = \frac{1 \cdot 2}{3 \cdot 2} = \frac{2}{6} = \frac{1 \cdot 5}{3 \cdot 5} = \frac{5}{15} \quad \text{etc.}$$

b) Dividing the numerator and denominator of a rational number by the same factor, which is called cancelling or reducing, does not change the value of the number.

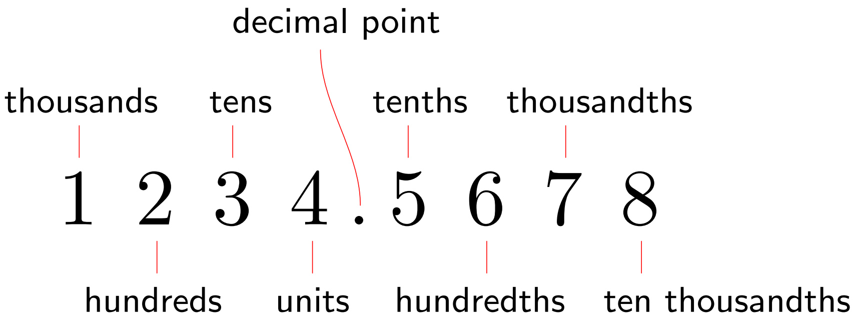
$$\frac{75}{105} = \frac{75/5}{105/5} = \frac{15}{21} = \frac{15/3}{21/3} = \frac{5}{7} \quad \text{etc.}$$

Irrational numbers

The numbers on the real-number axis that can not be written as a fraction are called irrational numbers. Examples of irrational numbers are most roots, for example: $\sqrt{2}$ and $\sqrt{3}$, but also numbers such as π.

Decimal form

All types of real numbers can be written in decimal form, with an arbitrary number of decimal places. Decimal integers written to the right of the decimal point specify the number of tenths, hundredths, thousandths and so on. In the same way as the integers to the left of the decimal point indicate the number of units, tens, hundreds and so on.



Example 4

$$1234.5678 = 1000 + 200 + 30 + 4 + \frac{5}{10} + \frac{6}{100} + \frac{7}{1000} + \frac{8}{10000}$$

A rational number can be written in decimal form by performing the division. Thus $\frac{3}{4}$ is the same as "3 divided by 4", i.e. 0.75.

Read about long division on wikipedia.

Example 5

a) $\frac{1}{2} = 0.5 = 0.5\underline{0}$

b) $\frac{1}{3} = 0.333333\ldots = 0.\underline{3}$

c) $\frac{5}{12} = 0.4166666\ldots = 0.41\underline{6}$

d) $\frac{1}{7} = 0.142857142857\ldots = 0.\underline{142857}$

(A bar below a single digit means that it is repeated; a bar below a block of digits mean the whole block is repeated.)

As you can see, the rational numbers above have a periodic decimal expansion; that is, the decimal expansion ends up with a finite block of digits that is repeated endlessly. This is true of all rational numbers and distinguishes them from the irrational numbers which do not have a periodic pattern in their decimal expansion. Conversely it is also true that all numbers with a periodic decimal expansion are rational.

Example 6

The numbers π and $\sqrt{2}$ are irrational and therefore have no periodic patterns in their decimal expansion.

a) $\pi = 3.141\,592\,653\,589\,793\,238\,462\,643\ \ldots$

b) $\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\ \ldots$

Example 7

a) $0.600\ \ldots = 0.6 = \frac{6}{10} = \frac{3}{5}$

b) $0.35 = \frac{35}{100} = \frac{7}{20}$

c) $0.0025 = \frac{25}{10\,000} = \frac{1}{400}$

Example 8

The number $x = 0.215151515\ \ldots$ is rational, because it has a periodic decimal expansion. We can write this rational number as a ratio of two integers as follows.

Multiply the number by 10 which moves the decimal point one step to the right.

$$10\,x = 2.151515\ \ldots$$

Multiply the number by $10 \cdot 10 \cdot 10 = 1000$ moving the decimal point three steps to the right

$$1000\,x = 215.1515\ \ldots$$

Now we see that $1000\,x$ and $10\,x$ have the same digits to the right of the decimal point, so the difference between the numbers

$$1000x - 10x = 215.1515\ \ldots - 2.151515\ \ldots,$$

must be an integer,

$$990x = 213.$$

So

$$x = \frac{213}{990} = \frac{71}{330}.$$

Rounding off

Since it is impractical to use long decimal expansions, one often rounds off a number to an appropriate number of decimal places. The standard practice is that if the next digit is 0, 1, 2, 3 or 4 we round down, whereas if it is 5, 6, 7, 8 or 9, we round up.

We use the symbol $\approx$ (is approximately equal to) to show that a rounding off has taken place.

Example 9

Rounding off to 3 decimal places:

a) $1.0004 \approx 1.000$

b) $0.9999 \approx 1.000$

c) $2.9994999 \approx 2.999$

d) $2.99950 \approx 3.000$

Example 10

Rounding off to 4 decimal places:

a) $\pi \approx 3.1416$

b) $\frac{2}{3} \approx 0.6667$

Comparing numbers

To indicate the relative size between numbers one uses the symbols $>$ (is greater than), $<$ (is less than) and $=$ (is equal to). The relative size between two numbers can be determined either by expressing the numbers in decimal form or by representing them, if possible, as fractions with a common denominator (this can only be done if both numbers are rational).

Example 11

a) Which is greater, $\frac{1}{3}$ or 0.33?

We have that $x = \frac{1}{3}$ and $y = 0{,}33 = \frac{33}{100}$ have the common denominator $3 \cdot$

$100 = 300$, and therefore

$$x = \frac{1}{3} = \frac{100}{300} \quad \text{and} \quad y = 0{,}33 = \frac{33}{100} = \frac{99}{300}.$$

Since $100 > 99$ we have that $\frac{100}{300} > \frac{99}{300}$ and therefore $x > y$.

Alternatively, you can see that $1/3 > 0.33$ as $1/3 = 0.3333\ldots > 0.33$.

b) Which number is the larger of $\frac{2}{5}$ and $\frac{3}{7}$?

Write the numbers with a common denominator, e.g. $5 \cdot 7 = 35$:

$$\frac{2}{5} = \frac{14}{35} \quad \text{and} \quad \frac{3}{7} = \frac{15}{35}.$$

Thus $\frac{3}{7} > \frac{2}{5}$ as $\frac{15}{35} > \frac{14}{35}$.

Study advice

Basic and final tests

After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that...

Many solutions are wrong because of copying errors or other simple errors, and not because your understanding of the question is wrong.

Reviews

For those of you who want to deepen your studies or need more detailed explanations consider the following references

- Learn more about arithmetic in the English Wikipedia
  ([`http://en.wikipedia.org/wiki/Arithmetic`)
- Who discovered zero? Read more in “The MacTutor History of Mathematics archive”
  (`http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Zero.html`)
- Long division
  (`http://www.mathsisfun.com/long_division.html`)
- Did you know that $0{,}999\ldots = 1$?
  (`http://en.wikipedia.org/wiki/0.999...`)

Useful web sites
How many colours are needed to colour a map? How many times does one need to shuffle a deck of cards? What is the greatest prime number? Are there any "lucky numbers"? What is the most beautiful number? Listen to the famous writer and mathematician Simon Singh, who among other things, tells about the magic numbers 4 and 7, about the prime numbers, about Kepler's piles and about the concept of zero.

- Listen to the BBC programmes "5 Numbers" (`http://www.bbc.co.uk/radio4/science/5numbers1.shtml`)
- Listen to the BBC programmes "Another 5 numbers" (`http://www.bbc.co.uk/radio4/science/another5.shtml`)

1.1 Exercises

Exercise 1.1:1

Work out (without the help of a calculator)

a) $3-7-4+6-5$

b) $3-(7-4)+(6-5)$

c) $3-(7-(4+6)-5)$

d) $3-(7-(4+6))-5$

Exercise 1.1:2

Simplify

a) $(3-(7-4))(6-5)$

b) $3-(((7-4)+6)-5)$

c) $3\cdot(-7)-4\cdot(6-5)$

d) $3\cdot(-7)-(4+6)/(-5)$

Exercise 1.1:3

Determine whether the following are natural numbers, integers, rational numbers or irrational numbers.

a) 8

b) -4

c) $8-4$

d) $4-8$

e) $8\cdot(-4)$

f) $(-8)\cdot(-4)$

Exercise 1.1:4

Determine whether the following are natural numbers, integers, rational numbers or irrational numbers.

a) $\dfrac{4}{-8}$

b) $\dfrac{-8}{-4}$

c) $\dfrac{\sqrt{2}}{3}$

d) $\left(\dfrac{4}{\sqrt{2}}\right)^2$

e) $-\pi$

f) $\pi+1$

Exercise 1.1:5

Arrange the following numbers in ascending order.

a) 2, $\frac{3}{5}$, $\frac{5}{3}$ and $\frac{7}{3}$

b) $-\frac{1}{2}$, $-\frac{1}{5}$, $-\frac{3}{10}$ and $-\frac{1}{3}$

c) $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{8}$ and $\frac{21}{34}$

Exercise 1.1:6

Give the decimal expansion of the following to three decimal places.

a) $\frac{7}{6}$

b) $\frac{9}{4}$

c) $\frac{2}{7}$

d) $\sqrt{2}$

Exercise 1.1:7

Which of the following numbers are rational? Express them as a fraction of two integers.

a) 3,14

b) 3,1416 1416 1416 ...

c) 0,2 001 001 001 ...

d) 0,10 100 1000 10000 1 ... (one 1:, one 0:, one 1:, two 0:s, one 1:, three 0:s etc.)

1.2 Fractional arithmetic

Contents:

- Addition and subtraction of fractions
- Multiplication and division of fractions

Learning outcomes:

After this section you should have learned to:

- Calculate the value of expressions containing fractions, the four arithmetic operations and parentheses.
- Cancel down fractions as far as possible (reduction).
- Determine the lowest common denominator (LCD).

Fraction modification

A rational number can be written in many ways depending on the denominator one chooses to use. For example, we have that

$$0{,}25 = \frac{25}{100} = \frac{1}{4} = \frac{2}{8} = \frac{3}{12} = \frac{4}{16} \quad \text{etc.}$$

The value of a rational number is not changed by multiplying or dividing the numerator and denominator by the same number. The division operation is called reduction or cancellation.

Example 1

Multiplying numerator and denominator by the same number:

a) $\dfrac{2}{3} = \dfrac{2 \cdot 5}{3 \cdot 5} = \dfrac{10}{15}$

b) $\dfrac{5}{7} = \dfrac{5 \cdot 4}{7 \cdot 4} = \dfrac{20}{28}$

Dvinding numerator and denominator by the same number: (reducing or cancelling):

c) $\frac{9}{12} = \frac{9/3}{12/3} = \frac{3}{4}$

d) $\frac{72}{108} = \frac{72/2}{108/2} = \frac{36}{54} = \frac{36/6}{54/6} = \frac{6}{9} = \frac{6/3}{9/3} = \frac{2}{3}$

We usually specify a fraction in a form where cancellation has been performed as far as possible: this is called expressing it in its lowest terms. This can be laborious when large numbers are involved which is why, in long calculations, you should usually cancel as you go along.

Addition and subtraction of fractions

Fractions can only be added or subtracted if they have the same denominator. If they do not, they must each first be "top-and-bottom" multiplied in such a way that they do. This is called placing over a common denominator.

Example 2

a) $\frac{3}{5} + \frac{2}{3} = \frac{3 \cdot 3}{5 \cdot 3} + \frac{2 \cdot 5}{3 \cdot 5} = \frac{9}{15} + \frac{10}{15} = \frac{9+10}{15} = \frac{19}{15}$

b) $\frac{5}{6} - \frac{2}{9} = \frac{5 \cdot 3}{6 \cdot 3} - \frac{2 \cdot 2}{9 \cdot 2} = \frac{15}{18} - \frac{4}{18} = \frac{15-4}{18} = \frac{11}{18}$

A common denominator can always be found by simply multiplying the denominators of the two fractions together. Often, however, a smaller one can be found. The ideal is to find the lowest common denominator (LCD).

Example 3

a) $\frac{7}{15} - \frac{1}{12} = \frac{7 \cdot 12}{15 \cdot 12} - \frac{1 \cdot 15}{12 \cdot 15} = \frac{84}{180} - \frac{15}{180} = \frac{69}{180} = \frac{69/3}{180/3} = \frac{23}{60}$

b) $\frac{7}{15} - \frac{1}{12} = \frac{7 \cdot 4}{15 \cdot 4} - \frac{1 \cdot 5}{12 \cdot 5} = \frac{28}{60} - \frac{5}{60} = \frac{23}{60}$

c) $$\begin{aligned} \frac{1}{8} + \frac{3}{4} - \frac{1}{6} &= \frac{1 \cdot 4 \cdot 6}{8 \cdot 4 \cdot 6} + \frac{3 \cdot 8 \cdot 6}{4 \cdot 8 \cdot 6} - \frac{1 \cdot 8 \cdot 4}{6 \cdot 8 \cdot 4} \\ &= \frac{24}{192} + \frac{144}{192} - \frac{32}{192} = \frac{136}{192} = \frac{136/8}{192/8} = \frac{17}{24} \end{aligned}$$

d) $\dfrac{1}{8}+\dfrac{3}{4}-\dfrac{1}{6}=\dfrac{1\cdot 3}{8\cdot 3}+\dfrac{3\cdot 6}{4\cdot 6}-\dfrac{1\cdot 4}{6\cdot 4}=\dfrac{3}{24}+\dfrac{18}{24}-\dfrac{4}{24}=\dfrac{17}{24}$

If the denominators are of reasonable size, you can usually find the LCD by inspection. More generally, you can use prime factorisation.

Example 4

a) Simplify $\dfrac{1}{60}+\dfrac{1}{42}$.

Our aim is to find the smallest number that both 60 and 42 go into. First, decompose 60 and 42 into their prime factors.

$$60 = 2\cdot 2\cdot 3\cdot 5$$
$$42 = 2\cdot 3\cdot 7$$

For each prime factor, choose the larger power:

$$\text{LCD} = 2\cdot 2\cdot 3\cdot 5\cdot 7 = 420.$$

We then can write

$$\frac{1}{60}+\frac{1}{42}=\frac{1\cdot 7}{60\cdot 7}+\frac{1\cdot 2\cdot 5}{42\cdot 2\cdot 5}=\frac{7}{420}+\frac{10}{420}=\frac{17}{420}.$$

b) Simplify $\dfrac{2}{15}+\dfrac{1}{6}-\dfrac{5}{18}$.

Here,

$$\left.\begin{aligned} 15 &= 3\cdot 5\\ 6 &= 2\cdot 3\\ 18 &= 2\cdot 3\cdot 3 \end{aligned}\right\} \quad\Rightarrow\quad \text{LCD} = 2\cdot 3\cdot 3\cdot 5 = 90\,.$$

We then can write

$$\frac{2}{15}+\frac{1}{6}-\frac{5}{18}=\frac{2\cdot 2\cdot 3}{15\cdot 2\cdot 3}+\frac{1\cdot 3\cdot 5}{6\cdot 3\cdot 5}-\frac{5\cdot 5}{18\cdot 5}=\frac{12}{90}+\frac{15}{90}-\frac{25}{90}=\frac{2}{90}=\frac{1}{45}.$$

Multiplication

When a fraction is multiplied by an integer, only the numerator is multiplied; for example, it is obvious that $\frac{1}{3}$ multiplied by 2 is equal to $\frac{2}{3}$.

If two fractions are multiplied together, then the numerators are multiplied together and the denominators are multiplied together.

Example 5

a) $8 \cdot \frac{3}{7} = \frac{8 \cdot 3}{7} = \frac{24}{7}$

b) $\frac{2}{3} \cdot \frac{1}{5} = \frac{2 \cdot 1}{3 \cdot 5} = \frac{2}{15}$

Before doing a multiplication you should always check whether you can cancel first. Note that you can cancel on both sides of the multiplication sign.

Example 6

Compare the calculations:

a) $\frac{3}{5} \cdot \frac{2}{3} = \frac{3 \cdot 2}{5 \cdot 3} = \frac{6}{15} = \frac{6/3}{15/3} = \frac{2}{5}$

b) $\frac{3}{5} \cdot \frac{2}{3} = \frac{\cancel{3} \cdot 2}{5 \cdot \cancel{3}} = \frac{2}{5}$

In 6b the 3 has been cancelled at an earlier stage than in 6a.

Example 7

a) $\frac{7}{10} \cdot \frac{2}{7} = \frac{\cancel{7}}{10} \cdot \frac{2}{\cancel{7}} = \frac{1}{10} \cdot \frac{2}{1} = \frac{1}{\cancel{2} \cdot 5} \cdot \frac{\cancel{2}}{1} = \frac{1}{5} \cdot \frac{1}{1} = \frac{1}{5}$

b) $\frac{14}{15} \cdot \frac{20}{21} = \frac{2 \cdot 7}{3 \cdot 5} \cdot \frac{4 \cdot 5}{3 \cdot 7} = \frac{2 \cdot \cancel{7}}{3 \cdot 5} \cdot \frac{4 \cdot 5}{3 \cdot \cancel{7}} = \frac{2}{3 \cdot \cancel{5}} \cdot \frac{4 \cdot \cancel{5}}{3} = \frac{2}{3} \cdot \frac{4}{3} = \frac{2 \cdot 4}{3 \cdot 3} = \frac{8}{9}$

Division

If $\frac{1}{4}$ is divided by 2 one gets the answer $\frac{1}{8}$. If $\frac{1}{2}$ is divided by 5 one gets the result $\frac{1}{10}$. We have that

$$\frac{\frac{1}{4}}{2} = \frac{1}{4 \cdot 2} = \frac{1}{8} \qquad \text{and} \qquad \frac{\frac{1}{2}}{5} = \frac{1}{2 \cdot 5} = \frac{1}{10}.$$

When a fraction is divided by an integer the denominator is multiplied by the integer.

Example 8

a) $\frac{3}{5}\Big/4 = \frac{3}{5 \cdot 4} = \frac{3}{20}$

b) $\frac{6}{7}\Big/3 = \frac{6}{7 \cdot 3} = \frac{2 \cdot \cancel{3}}{7 \cdot \cancel{3}} = \frac{2}{7}$

More generally, when a number is divided by a fraction the number is multiplied by the same fraction, inverted (that is, upside down). For example, dividing by $\frac{1}{2}$ is the same as multiplying by $\frac{2}{1}$; that is, by 2.

Example 9

a) $\dfrac{3}{\frac{1}{2}} = 3 \cdot \frac{2}{1} = \frac{3 \cdot 2}{1} = 6$

b) $\dfrac{5}{\frac{3}{7}} = 5 \cdot \frac{7}{3} = \frac{5 \cdot 7}{3} = \frac{35}{3}$

c) $\dfrac{\frac{2}{3}}{\frac{5}{8}} = \frac{2}{3} \cdot \frac{8}{5} = \frac{2 \cdot 8}{3 \cdot 5} = \frac{16}{15}$

d) $\dfrac{\frac{3}{4}}{\frac{9}{10}} = \frac{3}{4} \cdot \frac{10}{9} = \frac{\cancel{3}}{2 \cdot \cancel{2}} \cdot \frac{\cancel{2} \cdot 5}{\cancel{3} \cdot 3} = \frac{5}{2 \cdot 3} = \frac{5}{6}$

Why is it that division by a fraction is the same as multiplication by the same fraction, upside down? The explanation is that if a fraction is multiplied by "itself upside down", the product is always 1. For example,

$$\frac{2}{3}\cdot\frac{3}{2}=\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}=1 \qquad \text{oder} \qquad \frac{9}{17}\cdot\frac{17}{9}=\frac{\cancel{9}}{\cancel{17}}\cdot\frac{\cancel{17}}{\cancel{9}}=1.$$

If in a division of fractions one multiplies the numerator and denominator with the inverse of the denominator then the resulting fraction will have denominator 1. Thus the result is the numerator multiplied by the inverse of the original denominator.

Example 10

$$\frac{\frac{2}{3}}{\frac{5}{7}}=\frac{\frac{2}{3}\cdot\frac{7}{5}}{\frac{5}{7}\cdot\frac{7}{5}}=\frac{\frac{2}{3}\cdot\frac{7}{5}}{1}=\frac{2}{3}\cdot\frac{7}{5}$$

Fractions as a proportion of a whole

Rational numbers are numbers that can be writen as fractions, they can subsequently be converted to decimal form or be marked on a real-number axis. In our everyday language they are also used to describe the proportion of something. Below are given some examples. Note how we use the word "of", which can lead to a multiplication or a division.

Example 11

a) Jack invested 20 € and Jill 50 €.

Jack's share is $\frac{20}{50+20}=\frac{20}{70}=\frac{2}{7}$ and he must be given $\frac{2}{7}$ of the profits.

b) What proportion is 45 € of 100 €?

Answer: 45 € is $\frac{45}{100}=\frac{9}{20}$ of 100 €.

c) What proportion is $\frac{1}{3}$ litres of $\frac{1}{2}$ litre?

Answer: $\frac{1}{3}$ litres is $\frac{\frac{1}{3}}{\frac{1}{2}}=\frac{1}{3}\cdot\frac{2}{1}=\frac{2}{3}$ litres.

d) How much is $\frac{5}{8}$ of 1000?

Answer: $\frac{5}{8}\cdot 1000=\frac{5000}{8}=625$.

e) How much is $\frac{2}{3}$ of $\frac{6}{7}$?

Answer: $\frac{2}{3} \cdot \frac{6}{7} = \frac{2}{\cancel{3}} \cdot \frac{2 \cdot \cancel{3}}{7} = \frac{2 \cdot 2}{7} = \frac{4}{7}$.

Mixed expressions

When fractions appear in calculations one must follow the usual methods for arithmetic operations and their priority (multiplication/division before addition/subtraction). Remember also that the numerator and denominator involved in a division are calculated separately before the division is performed ("invisible parentheses").

Example 12

a) $$\frac{1}{\frac{2}{3}+\frac{3}{4}} = \frac{1}{\frac{2\cdot 4}{3\cdot 4}+\frac{3\cdot 3}{4\cdot 3}} = \frac{1}{\frac{8}{12}+\frac{9}{12}} = \frac{1}{\frac{17}{12}} = 1\cdot\frac{12}{17} = \frac{12}{17}$$

b) $$\frac{\frac{4}{3}-\frac{1}{6}}{\frac{4}{3}+\frac{1}{6}} = \frac{\frac{4\cdot 2}{3\cdot 2}-\frac{1}{6}}{\frac{4\cdot 2}{3\cdot 2}+\frac{1}{6}} = \frac{\frac{8}{6}-\frac{1}{6}}{\frac{8}{6}+\frac{1}{6}} = \frac{\frac{7}{6}}{\frac{9}{6}} = \frac{7}{\cancel{6}}\cdot\frac{\cancel{6}}{9} = \frac{7}{9}$$

c) $$\frac{3-\frac{3}{5}}{\frac{2}{3}-2} = \frac{\frac{3\cdot 5}{5}-\frac{3}{5}}{\frac{2}{3}-\frac{2\cdot 3}{3}} = \frac{\frac{15}{5}-\frac{3}{5}}{\frac{2}{3}-\frac{6}{3}} = \frac{\frac{12}{5}}{-\frac{4}{3}} = \frac{12}{5}\cdot\left(-\frac{3}{4}\right)$$

$$= -\frac{3\cdot\cancel{4}}{5}\cdot\frac{3}{\cancel{4}} = -\frac{3\cdot 3}{5} = -\frac{9}{5}$$

d) $$\frac{\frac{1}{\frac{1}{2}+\frac{1}{3}}-\frac{3}{5}\cdot\frac{1}{3}}{\frac{2}{3}\Big/\frac{1}{5}-\frac{\frac{1}{4}-\frac{1}{3}}{2}} = \frac{\frac{1}{\frac{3}{6}+\frac{2}{6}}-\frac{3\cdot 1}{5\cdot 3}}{\frac{2}{3}\cdot\frac{5}{1}-\frac{\frac{3}{12}-\frac{4}{12}}{2}} = \frac{\frac{1}{\frac{5}{6}}-\frac{1}{5}}{\frac{10}{3}-\frac{-\frac{1}{12}}{2}} = \frac{\frac{6}{5}-\frac{1}{5}}{\frac{10}{3}+\frac{1}{24}}$$

$$= \frac{1}{\frac{80}{24}+\frac{1}{24}} = \frac{1}{\frac{81}{24}} = \frac{24}{81} = \frac{8}{27}$$

Study advice

Basic and final tests
After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that:
You should try to write an expression in the simplest possible terms. What is the "simplest" depends usually on the context.

It is important that you really master calculations with fractions. You should be able to find a common denominator and multiply or divide numerators and denominators by suitable numbers. These principles are basic when you have to calculate a rational expression and you will need them when you have to deal with other mathematical expressions and operations.

Reviews
For those of you who want to deepen your studies or need more detailed explanations consider the following references:

- Learn more about the fractions and calculating with fractions in the English Wikipedia
(`http://en.wikipedia.org/wiki/Fraction_(mathematics)`)

Useful web sites

- Experimenting interactively with fractions
(`http://nlvm.usu.edu/en/nav/frames_asid_105_g_2_t_1.html`)
- Play the prime number canon
(`http://www.math.kth.se/~gunnarj/BIENNALEN/fall4.html`)

1.2 Exercises

Exercise 1.2:1

Express as a single fraction

a) $\dfrac{7}{4}+\dfrac{11}{7}$ b) $\dfrac{2}{7}-\dfrac{1}{5}$ c) $\dfrac{1}{6}-\dfrac{2}{5}$

d) $\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{5}$ e) $\dfrac{8}{7}+\dfrac{3}{4}-\dfrac{4}{3}$

Exercise 1.2:2

Determine the lowest common denominator of

a) $\dfrac{1}{6}+\dfrac{1}{10}$ b) $\dfrac{1}{4}-\dfrac{1}{8}$ c) $\dfrac{1}{12}-\dfrac{1}{14}$ d) $\dfrac{2}{45}+\dfrac{1}{75}$

Exercise 1.2:3

Calculate the following by using the lowest common denominator.

a) $\dfrac{3}{20}+\dfrac{7}{50}-\dfrac{1}{10}$ b) $\dfrac{1}{24}+\dfrac{1}{40}-\dfrac{1}{16}$

Exercise 1.2:4

Simplify the following by writing each part as one fraction. The fraction should be in the simplest possible form.

a) $\dfrac{\dfrac{3}{5}}{\dfrac{7}{10}}$ b) $\dfrac{\dfrac{2}{7}}{\dfrac{3}{8}}$ c) $\dfrac{\dfrac{1}{4}-\dfrac{1}{5}}{\dfrac{3}{10}}$

Exercise 1.2:5

Simplify the following by writing each part as one fraction. The fraction should be in the simplest possible form.

a) $\dfrac{2}{\dfrac{1}{7}-\dfrac{1}{15}}$ b) $\dfrac{\dfrac{1}{2}+\dfrac{1}{3}}{\dfrac{1}{3}-\dfrac{1}{2}}$ c) $\dfrac{\dfrac{3}{10}-\dfrac{1}{5}}{\dfrac{7}{8}-\dfrac{3}{16}}$

Exercise 1.2:6

Simplify $\dfrac{\dfrac{2}{3+\dfrac{1}{2}}+\dfrac{\dfrac{1}{2}}{\dfrac{1}{4}-\dfrac{1}{3}}}{\dfrac{1}{2}-\dfrac{3}{2-\dfrac{2}{7}}}$.

1.3 Powers

Contents:

- Positive integer exponent
- Negative integer exponent
- Rational exponents
- Laws of exponents

Learning outcomes:

After this section you will have learned to:

- Recognise the concepts of base and exponent.
- Calculate integer power expressions.
- Use the laws of indices to simplify expressions containing powers.
- Know when the laws of indices are applicable.
- Determine which of two powers is the larger based on a comparison of the base or exponent/index.

Integer exponents

We use the multiplication symbol as a shorthand for repeated addition of the same number. For example:

$$4 + 4 + 4 + 4 + 4 = 4 \cdot 5.$$

In a similar way we use exponentials as a short-hand for repeated multiplication of the same number:

$$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^5.$$

The 4 is called the base of the power and the 5 is its exponent or index (pl. indices).

Example 1

a) $5^3 = 5 \cdot 5 \cdot 5 = 125$

b) $10^5 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 100000$

c) $0{,}1^3 = 0{,}1 \cdot 0{,}1 \cdot 0{,}1 = 0{,}001$

d) $(-2)^4 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) = 16,$ but
$-2^4 = -(2^4) = -(2 \cdot 2 \cdot 2 \cdot 2) = -16$

e) $2 \cdot 3^2 = 2 \cdot 3 \cdot 3 = 18,$ but $(2 \cdot 3)^2 = 6^2 = 36$

Example 2

a) $\left(\dfrac{2}{3}\right)^3 = \dfrac{2}{3} \cdot \dfrac{2}{3} \cdot \dfrac{2}{3} = \dfrac{2^3}{3^3} = \dfrac{8}{27}$

b) $(2 \cdot 3)^4 = (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 3) \cdot (2 \cdot 3) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 2^4 \cdot 3^4 = 1296$

The last example can be generalised to two useful rules when calculating powers:

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad \text{und} \quad (ab)^m = a^m b^m,$$

for b non-zero and m is a natural number.

Laws of exponents

There are a few more rules that lead on from the definition of power which are useful when performing calculations. You can see for example that

$$2^3 \cdot 2^5 = \underbrace{2 \cdot 2 \cdot 2}_{\text{3 factors}} \cdot \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{\text{5 factors}} = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{\text{(3+5) factors}} = 2^{3+5} = 2^8.$$

which can be expressed more generally as

$$a^m \cdot a^n = a^{m+n}.$$

There is also a useful simplification rule for the division of powers that have the same base.

$$\frac{2^7}{2^3} = \frac{2\cdot 2\cdot 2\cdot 2\cdot \cancel{2}\cdot \cancel{2}\cdot \cancel{2}}{\cancel{2}\cdot \cancel{2}\cdot \cancel{2}} = 2^{7-3} = 2^4.$$

The general rule is

$$\frac{a^m}{a^n} = a^{m-n}.$$

For the case when the base itself is a power there is another useful rule. We see that

$$(5^2)^3 = 5^2\cdot 5^2\cdot 5^2 = \underbrace{5\cdot 5}_{\text{2 factors}}\cdot\underbrace{5\cdot 5}_{\text{2 factors}}\cdot\underbrace{5\cdot 5}_{\text{2 factors}} = \underbrace{5\cdot 5\cdot 5\cdot 5\cdot 5\cdot 5}_{\text{3 times 2 factors}} = 5^{2\cdot 3} = 5^6$$

and

$$(5^3)^2 = 5^3\cdot 5^3 = \underbrace{5\cdot 5\cdot 5}_{\text{3 factors}}\cdot\underbrace{5\cdot 5\cdot 5}_{\text{3 factors}} = \underbrace{5\cdot 5\cdot 5\,\cdot\,5\cdot 5\cdot 5}_{\text{2 times 3 factors}} = 5^{3\cdot 2} = 5^6.$$

Generally, this can be written

$$(a^m)^n = a^{m\cdot n}.$$

Example 3

a) $2^9\cdot 2^{14} = 2^{9+14} = 2^{23}$

b) $5\cdot 5^3 = 5^1\cdot 5^3 = 5^{1+3} = 5^4$

c) $3^2\cdot 3^3\cdot 3^4 = 3^{2+3+4} = 3^9$

d) $10^5\cdot 1000 = 10^5\cdot 10^3 = 10^{5+3} = 10^8$

Example 4

a) $\dfrac{3^{100}}{3^{98}} = 3^{100-98} = 3^2$

b) $\dfrac{7^{10}}{7} = \dfrac{7^{10}}{7^1} = 7^{10-1} = 7^9$

Note that if a fraction has the same power expression in both the numerator and the denominator, then we can simplify in either of two ways:

$$\frac{5^3}{5^3} = 5^{3-3} = 5^0 \quad \text{as well as} \quad \frac{5^3}{5^3} = \frac{5 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5} = \frac{125}{125} = 1.$$

So the only way to ensure that the rules of exponents agree is if we make the following natural definition. For all non-zero a we have

$$a^0 = 1.$$

We can also run into examples where the exponent in the denominator is greater than that in the numerator. For example we have

$$\frac{3^4}{3^6} = 3^{4-6} = 3^{-2} \quad \text{and} \quad \frac{3^4}{3^6} = \frac{\cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3}}{\cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot 3 \cdot 3} = \frac{1}{3 \cdot 3} = \frac{1}{3^2}.$$

It is therefore necessary that we assume that

$$3^{-2} = \frac{1}{3^2}.$$

Once more in the interests of consistency, the general definition for negative exponents is that for all non zero numbers a, we have

$$a^{-n} = \frac{1}{a^n}.$$

Example 5

a) $\dfrac{7^{1293}}{7^{1293}} = 7^{1293-1293} = 7^0 = 1$

b) $3^7 \cdot 3^{-9} \cdot 3^4 = 3^{7+(-9)+4} = 3^2$

c) $0{,}001 = \dfrac{1}{1000} = \dfrac{1}{10^3} = 10^{-3}$

d) $0{,}008 = \dfrac{8}{1000} = \dfrac{1}{125} = \dfrac{1}{5^3} = 5^{-3}$

e) $\left(\frac{2}{3}\right)^{-1} = \frac{1}{\left(\frac{2}{3}\right)^{1}} = 1 \cdot \frac{3}{2} = \frac{3}{2}$

f) $\left(\frac{1}{3^2}\right)^{-3} = (3^{-2})^{-3} = 3^{(-2)\cdot(-3)} = 3^6$

g) $0{,}01^5 = (10^{-2})^5 = 10^{-2\cdot 5} = 10^{-10}$

If the base of a power is -1 then the expression will simplify to either -1 or $+1$ depending on the value of the exponent

$$
\begin{aligned}
&(-1)^1 = -1\\
&(-1)^2 = (-1)\cdot(-1) = +1\\
&(-1)^3 = (-1)\cdot(-1)^2 = (-1)\cdot 1 = -1\\
&(-1)^4 = (-1)\cdot(-1)^3 = (-1)\cdot(-1) = +1\\
&\quad \text{etc.}
\end{aligned}
$$

The rule is that $(-1)^n$ is equal to -1 if n is odd and equal to $+1$ if n is even.

Example 6

a) $(-1)^{56} = 1$ as 56 is an even number.

b) $\frac{1}{(-1)^{11}} = \frac{1}{-1} = -1$ as 11 is an odd integer.

c) $\frac{(-2)^{127}}{2^{130}} = \frac{(-1\cdot 2)^{127}}{2^{130}} = \frac{(-1)^{127}\cdot 2^{127}}{2^{130}} = \frac{-1\cdot 2^{127}}{2^{130}}$
$= -2^{127-130} = -2^{-3} = -\frac{1}{2^3} = -\frac{1}{8}$

Changing the base

A point to observe is that when simplifying expressions one should try if possible to combine powers by choosing the same base. This often involves selecting 2, 3 or 5 as a base. It is therefore a good idea to learn to recognise the smaller powers of these numbers, such as:

$$4 = 2^2,\ \ 8 = 2^3,\ \ 16 = 2^4,\ \ 32 = 2^5,\ \ 64 = 2^6,\ \ 128 = 2^7,\ \dots$$
$$9 = 3^2,\ \ 27 = 3^3,\ \ 81 = 3^4,\ \ 243 = 3^5,\ \dots$$
$$25 = 5^2,\ \ 125 = 5^3,\ \ 625 = 5^4,\ \dots$$

Similarly, one should become familiar with

$$\frac{1}{4} = \frac{1}{2^2} = 2^{-2},\ \ \frac{1}{8} = \frac{1}{2^3} = 2^{-3},\ \ \frac{1}{16} = \frac{1}{2^4} = 2^{-4},\ \dots$$
$$\frac{1}{9} = \frac{1}{3^2} = 3^{-2},\ \ \frac{1}{27} = \frac{1}{3^3} = 3^{-3},\ \dots$$
$$\frac{1}{25} = \frac{1}{5^2} = 5^{-2},\ \ \frac{1}{125} = \frac{1}{5^3} = 5^{-3},\ \dots$$

and so on.

Example 7

a) Write $8^3 \cdot 4^{-2} \cdot 16$ as a power with base 2.

$$\begin{aligned} 8^3 \cdot 4^{-2} \cdot 16 &= (2^3)^3 \cdot (2^2)^{-2} \cdot 2^4 = 2^{3\cdot 3} \cdot 2^{2\cdot(-2)} \cdot 2^4 \\ &= 2^9 \cdot 2^{-4} \cdot 2^4 = 2^{9-4+4} = 2^9 \end{aligned}$$

b) Write $\dfrac{27^2 \cdot (1/9)^{-2}}{81^2}$ as a power with base 3.

$$\begin{aligned} \frac{27^2 \cdot (1/9)^{-2}}{81^2} &= \frac{(3^3)^2 \cdot (1/3^2)^{-2}}{(3^4)^2} = \frac{(3^3)^2 \cdot (3^{-2})^{-2}}{(3^4)^2} \\ &= \frac{3^{3\cdot 2} \cdot 3^{(-2)\cdot(-2)}}{3^{4\cdot 2}} = \frac{3^6 \cdot 3^4}{3^8} = \frac{3^{6+4}}{3^8} = \frac{3^{10}}{3^8} = 3^{10-8} = 3^2 \end{aligned}$$

c) Write $\dfrac{81 \cdot 32^2 \cdot (2/3)^2}{2^5 + 2^4}$ in as simple a form as possible.

$$\begin{aligned} \frac{81 \cdot 32^2 \cdot (2/3)^2}{2^5 + 2^4} &= \frac{3^4 \cdot (2^5)^2 \cdot \dfrac{2^2}{3^2}}{2^{4+1} + 2^4} = \frac{3^4 \cdot 2^{5\cdot 2} \cdot \dfrac{2^2}{3^2}}{2^4 \cdot 2^1 + 2^4} = \frac{3^4 \cdot 2^{10} \cdot \dfrac{2^2}{3^2}}{2^4 \cdot (2^1 + 1)} \\ &= \frac{\dfrac{3^4 \cdot 2^{10} \cdot 2^2}{3^2}}{2^4 \cdot 3} = \frac{3^4 \cdot 2^{10} \cdot 2^2}{3^2 \cdot 2^4 \cdot 3} = 3^{4-2-1} \cdot 2^{10+2-4} = 3^1 \cdot 2^8 = 3 \cdot 2^8 \end{aligned}$$

Rational exponents

What happens if a number is raised to a rational (that is, a fractional) exponent? Do the definitions and the rules we have used in the above calculations still hold?

For instance we note that

$$2^{1/2} \cdot 2^{1/2} = 2^{1/2+1/2} = 2^1 = 2.$$

That is, $2^{1/2}$ is the number that, when multiplied by itself, gives 2; in other words, $\sqrt{2}$.

Generally, we define

$$a^{1/2} = \sqrt{a}.$$

(We must assume that $a \geq 0$ since negative numbers do not have real square roots.)

We also see that, for example,

$$5^{1/3} \cdot 5^{1/3} \cdot 5^{1/3} = 5^{1/3+1/3+1/3} = 5^1 = 5.$$

which implies that $5^{1/3} = \sqrt[3]{5}$. This can be generalised to

$$a^{1/n} = \sqrt[n]{a}.$$

By combining this definition with one of our previous laws for exponents, namely $((a^m)^n = a^{m \cdot n})$, we have that for all $a \geq 0$, the following holds:

$$a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$$

or

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a}\,)^m.$$

Example 8

a) $27^{1/3} = \sqrt[3]{27} = 3$ since $3 \cdot 3 \cdot 3 = 27$

b) $1000^{-1/3} = \dfrac{1}{1000^{1/3}} = \dfrac{1}{(10^3)^{1/3}} = \dfrac{1}{10^{3 \cdot \frac{1}{3}}} = \dfrac{1}{10^1} = \dfrac{1}{10}$

c) $\dfrac{1}{\sqrt{8}} = \dfrac{1}{8^{1/2}} = \dfrac{1}{(2^3)^{1/2}} = \dfrac{1}{2^{3/2}} = 2^{-3/2}$

d) $\dfrac{1}{16^{-1/3}} = \dfrac{1}{(2^4)^{-1/3}} = \dfrac{1}{2^{-4/3}} = 2^{-(-4/3)} = 2^{4/3}$

Comparison of powers

If we do not have access to calculators and wish to compare the size of powers, we can sometimes do this by comparing bases or exponents.

If the base of a power is greater than 1 then the power increases as the exponent increases. On the other hand, if the base lies between 0 and 1 then the power decreases as the exponent grows.

Example 9

a) $3^{5/6} > 3^{3/4}$ because the base 3 is greater than 1 and the first exponent $5/6$ is greater than the second exponent $3/4$.

b) $3^{-3/4} > 3^{-5/6}$ as the base is greater than 1 and the exponents satisfy $-3/4 > -5/6$.

c) $0.3^5 < 0.3^4$ as the base 0.3 is between 0 and 1 and $5 > 4$.

If a power has a positive exponent it increases as the base increases. The opposite applies if the exponent is negative, that is to say the power decreases as the base increases.

Example 10

a) $5^{3/2} > 4^{3/2}$ as the base 5 is larger than the base 4 and both powers have the same positive exponent $3/2$.

b) $2^{-5/3} > 3^{-5/3}$ as the bases satisfy $2 < 3$ and the powers have a negative exponent $-5/3$.

Sometimes powers need to be rewritten in order to determine the relative sizes. For example to compare 125^2 with 36^3 we can rewrite them as

$$125^2 = (5^3)^2 = 5^6 \quad \text{and} \quad 36^3 = (6^2)^3 = 6^6$$

after which we see that $36^3 > 125^2$.

Example 11

For each of the following pairs of numbers, determine which is the greater:

a) $25^{1/3}$ and $5^{3/4}$.

The base 25 can be rewritten in terms of the second base 5 by putting $25 = 5 \times 5 = 5^2$. Therefore

$$25^{1/3} = (5^2)^{1/3} = 5^{2\cdot\frac{1}{3}} = 5^{2/3},$$

hence we see that

$$5^{3/4} > 25^{1/3}$$

since $\frac{3}{4} > \frac{2}{3}$ and the base 5 is larger than 1.

b) $(\sqrt{8}\,)^5$ and 128.

Both 8 and 128 can be written as powers of 2

$$\begin{aligned} 8 &= 2 \cdot 4 = 2 \cdot 2 \cdot 2 = 2^3, \\ 128 &= 2 \cdot 64 = 2 \cdot 2 \cdot 32 = 2 \cdot 2 \cdot 2 \cdot 16 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 8 \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2^3 = 2^7. \end{aligned}$$

This gives

$$\begin{aligned} (\sqrt{8}\,)^5 &= (8^{1/2})^5 = (8)^{5/2} = (2^3)^{5/2} = 2^{3\cdot\frac{5}{2}} = 2^{15/2}, \\ 128 &= 2^7 = 2^{14/2}, \end{aligned}$$

and thus

$$(\sqrt{8}\,)^5 > 128$$

because $\frac{15}{2} > \frac{14}{2}$ and the base 2 is greater than 1.

c) $(8^2)^{1/5}$ and $(\sqrt{27}\,)^{4/5}$.

Since $8 = 2^3$ and $27 = 3^3$, the first step is to simplify and write the numbers as powers of 2 and 3 respectively,

$$\begin{aligned} (8^2)^{1/5} &= (8)^{2/5} = (2^3)^{2/5} = 2^{3\cdot\frac{2}{5}} = 2^{6/5}, \\ (\sqrt{27}\,)^{4/5} &= (27^{1/2})^{4/5} = 27^{\frac{1}{2}\cdot\frac{4}{5}} = 27^{2/5} = (3^3)^{2/5} = 3^{3\cdot\frac{2}{5}} = 3^{6/5}. \end{aligned}$$

Now we see that

$$(\sqrt{27}\,)^{4/5} > (8^2)^{1/5}$$

because $3 > 2$ and exponent $\frac{6}{5}$ is positive.

d) $3^{1/3}$ and $2^{1/2}$

We rewrite the exponents due to them having a common denominator

$$\frac{1}{3} = \frac{2}{6} \quad \text{and} \quad \frac{1}{2} = \frac{3}{6}.$$

This gives

$$\begin{aligned} 3^{1/3} &= 3^{2/6} = (3^2)^{1/6} = 9^{1/6}, \\ 2^{1/2} &= 2^{3/6} = (2^3)^{1/6} = 8^{1/6}, \end{aligned}$$

and we see that

$$3^{1/3} > 2^{1/2}$$

because $9 > 8$ and the exponent $1/6$ is positive.

Study advice

Basic and final tests
After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that...
The number raised to the power 0 is always 1 as long as the number (the base) is not 0.

Reviews
For those of you who want to deepen your studies or need more detailed explanations consider the following references

- Learn more about powers in the English Wikipedia (`http://en.wikipedia.org/wiki/Exponent`)
- What is the greatest prime number? Read more at The Prime Page (`http://primes.utm.edu/`)

Useful web sites

- Here you can practise the laws of exponents (`http://www.ltcconline.net/greenl/java/BasicAlgebra/ExponentRules/ExponentRules.html`)

1.3 Exercises

Exercise 1.3:1

Calculate

a) $2^3 \cdot 3^2$ b) $3^5 \cdot 9^{-2}$ c) $(-5)^3$ d) $\left(\frac{2}{3}\right)^{-3}$

Exercise 1.3:2

Write each of the following as a power of 2

a) $2 \cdot 4 \cdot 8$ b) $0{,}25$ c) 1

Exercise 1.3:3

Write each of the following as a power of 3

a) $\frac{1}{3}$ b) 243 c) 9^2 d) $\frac{1}{27}$ e) $\frac{3}{9^2}$

Exercise 1.3:4

Calculate

a) $2^9 \cdot 2^{-7}$ b) $3^{13} \cdot 9^{-3} \cdot 27^{-2}$ c) $\frac{5^{12}}{5^{-4}} \cdot (5^2)^{-6}$

d) $2^{2^3} \cdot (-2)^{-4}$ e) $625 \cdot (5^8 + 5^9)^{-1}$

Exercise 1.3:5

Calculate

a) $4^{1/2}$ b) $4^{-1/2}$

c) $9^{3/2}$ d) $\left(47^{2/3}\right)^3$

e) $3^{1{,}4} \cdot 3^{0{,}6}$ f) $\left(125^{1/3}\right)^2 \cdot \left(27^{1/3}\right)^{-2} \cdot 9^{1/2}$

Exercise 1.3:6

In each of the following cases, determine which is the larger of the two numbers.

a) $256^{1/3}$ and $200^{1/3}$ b) $0{,}5^{-3}$ and $0{,}4^{-3}$ c) $0{,}2^5$ and $0{,}2^7$

d) $400^{1/3}$ and $\left(5^{1/3}\right)^4$ e) $125^{1/2}$ and $625^{1/3}$ f) 2^{56} and 3^{40}

2.1 Algebraic expressions

Contents:

- Distributive law
- Expansion and factorisation
- Difference of two squares
- Rational expressions

Learning outcomes:

After this section you will have learned how to:

- Simplify complicated algebraic expressions.
- Factorise expressions, including perfect squares and the difference of two squares.
- Expand expressions, including perfect squares and the difference of two squares.

Distributive Law

The distributive law specifies how to multiply a bracketed expression by a factor.

$$a(b+c) = ab + ac$$

Example 1

a) $4(x+y) = 4x + 4y$

b) $2(a-b) = 2a - 2b$

c) $x\left(\frac{1}{x} + \frac{1}{x^2}\right) = x \cdot \frac{1}{x} + x \cdot \frac{1}{x^2} = \frac{\cancel{x}}{\cancel{x}} + \frac{\cancel{x}}{x^{\cancel{2}}} = 1 + \frac{1}{x}$

d) $a(x+y+z) = ax + ay + az$

Using the distributive law we can also see how to tackle a minus sign in front of a bracketed expression. The rule says that a minus sign in front of a bracket can be eliminated if all the terms inside the brackets switch signs.

Example 2

a) $-(x+y) = (-1)\cdot(x+y) = (-1)x + (-1)y = -x - y$

b) $-(x^2 - x) = (-1)\cdot(x^2 - x) = (-1)x^2 - (-1)x = -x^2 + x$
where we have in the final step used $-(-1)x = (-1)(-1)x = 1\cdot x = x$.

c) $$\begin{aligned} -(x+y-y^3) &= (-1)\cdot(x+y-y^3) = (-1)\cdot x + (-1)\cdot y - (-1)\cdot y^3 \\ &= -x - y + y^3 \end{aligned}$$

d) $$\begin{aligned} x^2 - 2x - (3x+2) &= x^2 - 2x - 3x - 2 = x^2 - (2+3)x - 2 \\ &= x^2 - 5x - 2 \end{aligned}$$

Applying the distributive law this way round - converting a product of factors into a sum of terms — is called *expanding*. If the distributive law is applied in reverse we say we “factorise” the expression. We usually want to factorise as thoroughly as possible, by identifying the highet factor shared by all the terms.

Example 3

a) $3x + 9y = 3x + 3\cdot 3y = 3(x+3y)$

b) $xy + y^2 = xy + y\cdot y = y(x+y)$

c) $2x^2 - 4x = 2x\cdot x - 2\cdot 2\cdot x = 2x(x-2)$

d) $\dfrac{y-x}{x-y} = \dfrac{-(x-y)}{x-y} = \dfrac{-1}{1} = -1$

Squaring

On occasions the distributive law has to be used repeatedly to deal with larger expressions. If we consider

$$(a+b)(c+d)$$

and regard $a+b$ as a factor that multiplies the bracketed expression $(c+d)$ we get

$$\begin{aligned} \,(c+d) &= \,c + \,d, \\ (a+b)\,(c+d) &= (a+b)\,c + (a+b)\,d. \end{aligned}$$

Then the c and the d are multiplied into their respective brackets,

$$(a+b)c + (a+b)d = ac + bc + ad + bd.$$

A mnemonic for this formula is:

$$(a+b)(c+d) = ac + ad + bc + bd$$

Example 4

a) $\begin{aligned}[t] (x+1)(x-2) &= x \cdot x + x \cdot (-2) + 1 \cdot x + 1 \cdot (-2) = x^2 - 2x + x - 2 \\ &= x^2 - x - 2 \end{aligned}$

b) $\begin{aligned}[t] 3(x-y)(2x+1) &= 3(x \cdot 2x + x \cdot 1 - y \cdot 2x - y \cdot 1) = 3(2x^2 + x - 2xy - y) \\ &= 6x^2 + 3x - 6xy - 3y \end{aligned}$

c) $\begin{aligned}[t] (1-x)(2-x) &= 1 \cdot 2 + 1 \cdot (-x) - x \cdot 2 - x \cdot (-x) = 2 - x - 2x + x^2 \\ &= 2 - 3x + x^2 \end{aligned}$

where we have used $-x \cdot (-x) = (-1)x \cdot (-1)x = (-1)^2 x^2 = 1 \cdot x^2 = x^2$.

Two important special cases of the above formula are when $a+b$ and $c+d$ are the same expression

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2 \\ (a-b)^2 &= a^2 - 2ab + b^2 \end{aligned}$$

Example 5

a) $(x+2)^2 = x^2 + 2 \cdot 2x + 2^2 = x^2 + 4x + 4$

b) $(-x+3)^2 = (-x)^2 + 2 \cdot 3(-x) + 3^2 = x^2 - 6x + 9$

where $(-x)^2 = ((-1)x)^2 = (-1)^2 x^2 = 1 \cdot x^2 = x^2$.

c) $(x^2-4)^2 = (x^2)^2 - 2\cdot 4x^2 + 4^2 = x^4 - 8x^2 + 16$

d) $$\begin{aligned}(x+1)^2 - (x-1)^2 &= (x^2+2x+1) - (x^2-2x+1)\\ &= x^2+2x+1-x^2+2x-1\\ &= 2x+2x = 4x\end{aligned}$$

e) $$\begin{aligned}(2x+4)(x+2) &= 2(x+2)(x+2) = 2(x+2)^2 = 2(x^2+4x+4)\\ &= 2x^2+8x+8\end{aligned}$$

f) $$\begin{aligned}(x-2)^3 &= (x-2)(x-2)^2 = (x-2)(x^2-4x+4)\\ &= x\cdot x^2 + x\cdot(-4x) + x\cdot 4 - 2\cdot x^2 - 2\cdot(-4x) - 2\cdot 4\\ &= x^3 - 4x^2 + 4x - 2x^2 + 8x - 8 = x^3 - 6x^2 + 12x - 8\end{aligned}$$

These rules are also used in the reverse direction to factorise expressions.

Example 6

a) $x^2 + 2x + 1 = (x+1)^2$

b) $x^6 - 4x^3 + 4 = (x^3)^2 - 2\cdot 2x^3 + 2^2 = (x^3-2)^2$

c) $x^2 + x + \frac{1}{4} = x^2 + 2\cdot\frac{1}{2}x + \left(\frac{1}{2}\right)^2 = \left(x+\frac{1}{2}\right)^2$

Difference of two squares

A third special case of the first formula in the last section is the difference of two squares rule.

Difference of two squares:

$$(a+b)(a-b) = a^2 - b^2$$

This formula can be obtained directly by expanding the left hand side

$$(a+b)(a-b) = a\cdot a + a\cdot(-b) + b\cdot a + b\cdot(-b) = a^2 - ab + ab - b^2 = a^2 - b^2.$$

Example 7

a) $(x-4y)(x+4y)=x^2-(4y)^2=x^2-16y^2$

b) $(x^2+2x)(x^2-2x)=(x^2)^2-(2x)^2=x^4-4x^2$

c) $(y+3)(3-y)=(3+y)(3-y)=3^2-y^2=9-y^2$

d) $$\begin{aligned} x^4-16 &= (x^2)^2-4^2=(x^2+4)(x^2-4)=(x^2+4)(x^2-2^2) \\ &= (x^2+4)(x+2)(x-2) \end{aligned}$$

Rational expressions

Working with fractions containing algebraic expressions is very similar to carrying out ordinary calculations with fractions.

Multiplication and division of algebraic fractions follow the same rules that apply to ordinary fractions,

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d} \qquad \text{und} \qquad \frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a\cdot d}{b\cdot c}.$$

Example 8

a) $$\frac{3x}{x-y}\cdot\frac{4x}{2x+y}=\frac{3x\cdot 4x}{(x-y)\cdot(2x+y)}=\frac{12x^2}{(x-y)(2x+y)}$$

b) $$\frac{\frac{a}{x}}{\frac{x+1}{a}}=\frac{a^2}{x(x+1)}$$

c) $$\frac{\frac{x}{(x+1)^2}}{\frac{x-2}{x-1}}=\frac{x(x-1)}{(x-2)(x+1)^2}$$

A fractional expression can have its numerator and denominator multiplied by the same factor

$$\frac{x+2}{x+1} = \frac{(x+2)(x+3)}{(x+1)(x+3)} = \frac{(x+2)(x+3)(x+4)}{(x+1)(x+3)(x+4)} = \dots$$

We can cancel factors that the numerator and denominator have in common

$$\frac{(x+2)\cancel{(x+3)}(x+4)}{(x+1)\cancel{(x+3)}(x+4)} = \frac{(x+2)\cancel{(x+4)}}{(x+1)\cancel{(x+4)}} = \frac{x+2}{x+1}.$$

Example 9

a) $$\frac{x}{x+1} = \frac{x}{x+1} \cdot \frac{x+2}{x+2} = \frac{x(x+2)}{(x+1)(x+2)}$$

b) $$\frac{x^2-1}{x(x^2-1)} = \frac{1}{x}$$

c) $$\begin{aligned}\frac{(x^2-y^2)(x-2)}{(x^2-4)(x+y)} &= \{\text{ Difference of two squares }\}\\ &= \frac{(x+y)(x-y)(x-2)}{(x+2)(x-2)(x+y)} = \frac{x-y}{x+2}\end{aligned}$$

When fractional expressions are added or subtracted they may need to be placed over a common denominator.

$$\frac{1}{x} - \frac{1}{x-1} = \frac{1}{x} \cdot \frac{x-1}{x-1} - \frac{1}{x-1} \cdot \frac{x}{x} = \frac{x-1}{x(x-1)} - \frac{x}{x(x-1)} = \frac{x-1-x}{x(x-1)} = \frac{-1}{x(x-1)}.$$

As with ordinary fractions, it's possible to find a common denominator by simply multiplying the two denominators, but it is better to find the smallest, simplest expression that both denominators go into: the lowest common denominator or LCD.

Example 10

a) $\dfrac{1}{x+1} + \dfrac{1}{x+2}$ has LCD $= (x+1)(x+2)$.

Convert the first term using $(x+2)$ and the second term using $(x+1)$

$$\begin{aligned}\frac{1}{x+1} + \frac{1}{x+2} &= \frac{x+2}{(x+1)(x+2)} + \frac{x+1}{(x+2)(x+1)}\\ &= \frac{x+2+x+1}{(x+1)(x+2)} = \frac{2x+3}{(x+1)(x+2)}\,.\end{aligned}$$

b) $\dfrac{1}{x}+\dfrac{1}{x^2}$ has LCD $= x^2$.

We only need to convert the first term to get a common denominator

$$\frac{1}{x}+\frac{1}{x^2}=\frac{x}{x^2}+\frac{1}{x^2}=\frac{x+1}{x^2}.$$

c) $\dfrac{1}{x(x+1)^2}-\dfrac{1}{x^2(x+2)}$ has LCD $= x^2(x+1)^2(x+2)$.

The first term is converted using $x(x+2)$ while the other term is converted using $(x+1)^2$,

$$\begin{aligned}
\frac{1}{x(x+1)^2}-\frac{1}{x^2(x+2)} &= \frac{x(x+2)}{x^2(x+1)^2(x+2)}-\frac{(x+1)^2}{x^2(x+1)^2(x+2)}\\
&= \frac{x^2+2x}{x^2(x+1)^2(x+2)}-\frac{x^2+2x+1}{x^2(x+1)^2(x+2)}\\
&= \frac{x^2+2x-(x^2+2x+1)}{x^2(x+1)^2(x+2)}\\
&= \frac{x^2+2x-x^2-2x-1}{x^2(x+1)^2(x+2)}\\
&= \frac{-1}{x^2(x+1)^2(x+2)}.
\end{aligned}$$

d) $\dfrac{x}{x+1}-\dfrac{1}{x(x-1)}-1$ has $x(x-1)(x+1)$.

We must convert all the terms so that they have the common denominator $x(x-1)(x+1)$,

$$\begin{aligned}
\frac{x}{x+1}-\frac{1}{x(x-1)}-1 &= \frac{x^2(x-1)}{x(x-1)(x+1)}-\frac{x+1}{x(x-1)(x+1)}-\frac{x(x-1)(x+1)}{x(x-1)(x+1)}\\
&= \frac{x^3-x^2}{x(x-1)(x+1)}-\frac{x+1}{x(x-1)(x+1)}-\frac{x^3-x}{x(x-1)(x+1)}\\
&= \frac{x^3-x^2-(x+1)-(x^3-x)}{x(x-1)(x+1)}\\
&= \frac{x^3-x^2-x-1-x^3+x}{x(x-1)(x+1)}\\
&= \frac{-x^2-1}{x(x-1)(x+1)}.
\end{aligned}$$

To simplify large expressions it is often necessary both to cancel factors and to multiply numerators and denominators by factors. Where possible, therefore, we should keep expressions in a factorised form, and not expand expressions we will later need to factorise again.

Example 11

a) $$\frac{1}{x-2}-\frac{4}{x^2-4}=\frac{1}{x-2}-\frac{4}{(x+2)(x-2)}$$
$$=\{\text{ LCD}=(x+2)(x-2)\ \}$$
$$=\frac{x+2}{(x+2)(x-2)}-\frac{4}{(x+2)(x-2)}$$
$$=\frac{x+2-4}{(x+2)(x-2)}=\frac{x-2}{(x+2)(x-2)}=\frac{1}{x+2}$$

b) $$\frac{x+\dfrac{1}{x}}{x^2+1}=\frac{\dfrac{x^2}{x}+\dfrac{1}{x}}{x^2+1}=\frac{\dfrac{x^2+1}{x}}{x^2+1}=\frac{x^2+1}{x(x^2+1)}=\frac{1}{x}$$

c) $$\frac{\dfrac{1}{x^2}-\dfrac{1}{y^2}}{x+y}=\frac{\dfrac{y^2}{x^2y^2}-\dfrac{x^2}{x^2y^2}}{x+y}=\frac{\dfrac{y^2-x^2}{x^2y^2}}{x+y}=\frac{y^2-x^2}{x^2y^2(x+y)}$$
$$=\frac{(y+x)(y-x)}{x^2y^2(x+y)}=\frac{y-x}{x^2y^2}$$

Study advice

The basic and final tests
After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that...
If you make a mistake somewhere the rest of the calculation will be wrong, so be careful!

Use many intermediate steps. If you are unsure of a calculation do it in many small steps rather than one big step.

Do not expand unnecessarily. You later may be forced to factorise what you earlier expanded.

Reviews

- Learn more about algebra in the English Wikipedia (http://en.wikipedia.org/wiki/Algebra)
- Understanding Algebra (http://www.jamesbrennan.org/algebra/)

2.1 Exercises

Exercise 2.1:1

Expand

a)	$3x(x-1)$	b)	$(1+x-x^2)xy$	c)	$-x^2(4-y^2)$
d)	$x^3y^2\left(\dfrac{1}{y}-\dfrac{1}{xy}+1\right)$	e)	$(x-7)^2$	f)	$(5+4y)^2$
g)	$(y^2-3x^3)^2$	h)	$(5x^3+3x^5)^2$		

Exercise 2.1:2

Expand

a)	$(x-4)(x-5)-3x(2x-3)$	b)	$(1-5x)(1+15x)-3(2-5x)(2+5x)$
c)	$(3x+4)^2-(3x-2)(3x-8)$	d)	$(3x^2+2)(3x^2-2)(9x^4+4)$
e)	$(a+b)^2+(a-b)^2$		

Exercise 2.1:3

Factorise and simplify as much as possible

a)	x^2-36	b)	$5x^2-20$	c)	x^2+6x+9
d)	$x^2-10x+25$	e)	$18x-2x^3$	f)	$16x^2+8x+1$

Exercise 2.1:4

Determine the coefficients in front of x and x^2 when the following expressions are expanded out.

a) $(x+2)(3x^2-x+5)$

b) $(1+x+x^2+x^3)(2-x+x^2+x^4)$

c) $(x-x^3+x^5)(1+3x+5x^2)(2-7x^2-x^4)$

Exercise 2.1:5

Simplify as much as possible

a)	$\dfrac{1}{x-x^2}-\dfrac{1}{x}$	b)	$\dfrac{1}{y^2-2y}-\dfrac{2}{y^2-4}$
c)	$\dfrac{(3x^2-12)(x^2-1)}{(x+1)(x+2)}$	d)	$\dfrac{(y^2+4y+4)(2y-4)}{(y^2+4)(y^2-4)}$

Exercise 2.1:6

Simplify as much as possible

a) $\left(x-y+\dfrac{x^2}{y-x}\right)\left(\dfrac{y}{2x-y}-1\right)$ b) $\dfrac{x}{x-2}+\dfrac{x}{x+3}-2$

c) $\dfrac{2a+b}{a^2-ab}-\dfrac{2}{a-b}$ d) $\dfrac{a-b+\dfrac{b^2}{a+b}}{1-\left(\dfrac{a-b}{a+b}\right)^2}$

Exercise 2.1:7

Simplify the following by writing them as a single ordinary fraction

a) $\dfrac{2}{x+3}-\dfrac{2}{x+5}$ b) $x+\dfrac{1}{x-1}+\dfrac{1}{x^2}$ c) $\dfrac{ax}{a+1}-\dfrac{ax^2}{(a+1)^2}$

Exercise 2.1:8

Simplify the following fractions by writing them as a single ordinary

a) $\dfrac{\dfrac{x}{x+1}}{3+x}$ b) $\dfrac{\dfrac{3}{x}-\dfrac{1}{x}}{\dfrac{1}{x-3}}$ c) $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+x}}}$

2.2 Linear expressions

Contents:

- Linear equations
- Equation of a straight line
- Geometrical problems
- Regions that are defined using inequalities

Learning outcomes:

After this section you will have learned how to:

- Solve equations that are linear, or linear when simplified.
- Convert between the forms $y = mx + c$ and $ax + by + c = 0$.
- Sketch straight lines from their equations.
- Solve geometric problems that contain straight lines.
- Sketch regions defined by linear inequalities and determine the area of these regions.

First degree equations

To solve linear equations (also known as equations of degree one) we perform operations on both sides simultaneously, in such a way as to gradually simplify the equation and ultimately lead to x being alone on one side of it.

Example 1

a) Solve the equation $x + 3 = 7$.

Subtract 3 from both sides

$$x + 3 - 3 = 7 - 3.$$

The left-hand side then simplifies to x and we get

$$x = 7 - 3 = 4.$$

b) Solve the equation $3x = 6$.

Divide both sides by 3,

$$\frac{3x}{3} = \frac{6}{3}.$$

After cancelling 3 on the left-hand side we have

$$x = \frac{6}{3} = 2.$$

c) Solve the equation $2x + 1 = 5$.

First we subtract 1 from both sides to get $2x$ on its own on the left-hand side

$$2x = 5 - 1.$$

We then divide both sides by 2 to get the answer

$$x = \frac{4}{2} = 2.$$

A linear equation can always be written in the form $ax = b$. The solution is then simply $x = b/a$ (we must assume that $a \neq 0$). Sometimes, however, some work is needed in order to place the equation in this form. Here are a couple of examples.

Example 2

Solve the equation $2x - 3 = 5x + 7$.

Since x occurs on both the left and right hand sides we subtract $2x$ from both sides

$$2x - 3 - 2x = 5x + 7 - 2x,$$

and now x only appears on the right-hand side

$$-3 = 3x + 7.$$

We now subtract 7 from both sides

$$-3 - 7 = 3x + 7 - 7,$$

and get $3x$ on its own on the right-hand side

$$-10 = 3x.$$

The last step is to divide both sides by 3

$$\frac{-10}{3} = \frac{3x}{3},$$

giving

$$x = -\frac{10}{3}.$$

Example 3

Solve for x the equation $ax + 7 = 3x - b$.

By subtracting $3x$ from both sides

$$\begin{aligned} ax + 7 - 3x &= 3x - b - 3x, \\ ax + 7 - 3x &= - b, \end{aligned}$$

and then subtracting 7

$$\begin{aligned} ax + 7 - 3x - 7 &= -b - 7, \\ ax - 3x &= -b - 7. \end{aligned}$$

we have gathered together all the terms that contain x on the left-hand side and all other terms on the right-hand side. Since the terms on the left-hand side have x as a common factor x, they can be factored out

$$(a - 3)x = -b - 7.$$

Divide both sides with $a - 3$ giving

$$x = \frac{-b - 7}{a - 3}.$$

It is not always obvious that we are dealing with a linear equation. In the following two examples simplifications are needed to convert the original equation into a linear one.

Example 4

Solve the equation $(x - 3)^2 + 3x^2 = (2x + 7)^2$.

Expand the quadratic expressions on both sides

$$\begin{aligned} x^2 - 6x + 9 + 3x^2 &= 4x^2 + 28x + 49, \\ 4x^2 - 6x + 9 &= 4x^2 + 28x + 49. \end{aligned}$$

Subtract $4x^2$ from both sides

$$-6x + 9 = 28x + 49.$$

Add $6x$ to both sides

$$9 = 34x + 49.$$

Subtract 49 from both sides

$$-40 = 34x.$$

Divide both sides by 34

$$x = \frac{-40}{34} = -\frac{20}{17}.$$

Example 5

Solve the equation $\dfrac{x+2}{x^2+x} = \dfrac{3}{2+3x}$.

Collect both terms to one side

$$\frac{x+2}{x^2+x} - \frac{3}{2+3x} = 0.$$

Convert the terms so that they have the same denominator

$$\frac{(x+2)(2+3x)}{(x^2+x)(2+3x)} - \frac{3(x^2+x)}{(2+3x)(x^2+x)} = 0,$$

and simplify the numerator

$$\frac{(x+2)(2+3x) - 3(x^2+x)}{(x^2+x)(2+3x)} = 0,$$

$$\frac{3x^2+8x+4-(3x^2+3x)}{(x^2+x)(2+3x)} = 0,$$

$$\frac{5x+4}{(x^2+x)(2+3x)} = 0.$$

This equation is only satisfied when the numerator is equal to zero (whilst the denominator is not equal to zero);

$$5x + 4 = 0,$$

which gives that $x = -\frac{4}{5}$.

Straight lines

Functions such as

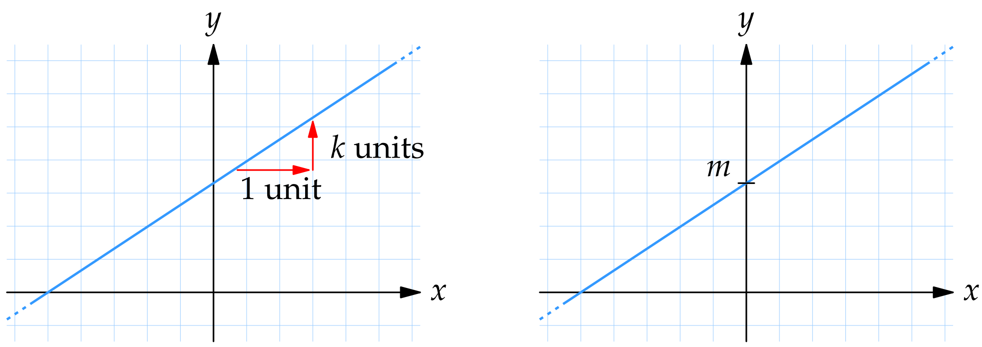
$$
\begin{aligned}
y &= 2x + 1, \\
y &= -x + 3 \quad \text{und} \\
y &= \tfrac{1}{2}x - 5
\end{aligned}
$$

are examples of linear functions. They have the general form

$$y = kx + m$$

where m and c are constants.

The graph of a linear function is always a straight line. The constant m indicates the gradient of the line with respect to the x-axis and c gives the intercept: that is, the y-coordinate of the point where the line intersects the y-axis.



The line $y = mx + c$ has gradient m and cuts the y-axis at $(0, c)$.

The constant m is called the gradient. To state that a straight line has gradient m means that for each increase of one unit in the positive x-direction there is an increase of m units in the positive y-direction. The gradient can thus be described as the "height of the unit step". Note that a negative gradient means that y decreases as x increases: that is, if

- $m > 0$ the line slopes upwards,
- $m < 0$ the line slopes downwards.

For a horizontal line (parallel to the x-axis) $m = 0$, whereas a vertical line (parallel to the y-axis) does not have a m value (a vertical line cannot be written in the form $y = mx + c$).

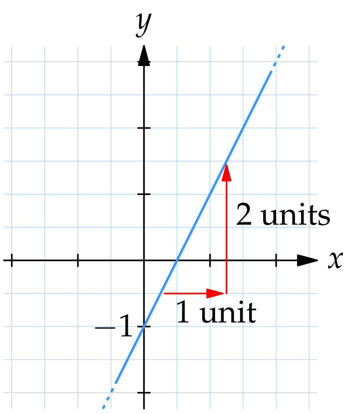
Example 6

a) Sketch the line $y = 2x - 1$.

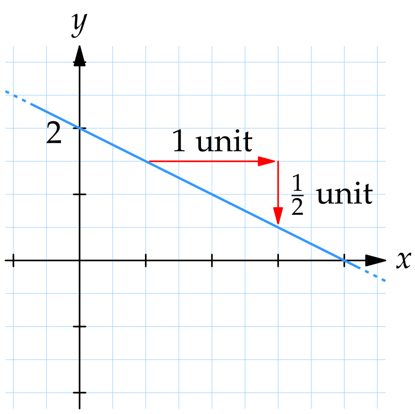
Comparing with the standard equation $y = mx + c$, we see that $m = 2$ and $c = -1$. This tells us that the line's gradient is 2 and that it cuts the y-axis at $(0, -1)$. See the figure below-left.

b) Sketch the line $y = 2 - \frac{1}{2}x$.

The equation of the line can be written as $y = -\frac{1}{2}x + 2$. From this we see that has the gradient $m = -\frac{1}{2}$and that the intercept $c = 2$. See the figure below to the right.



Line $y = 2x - 1$



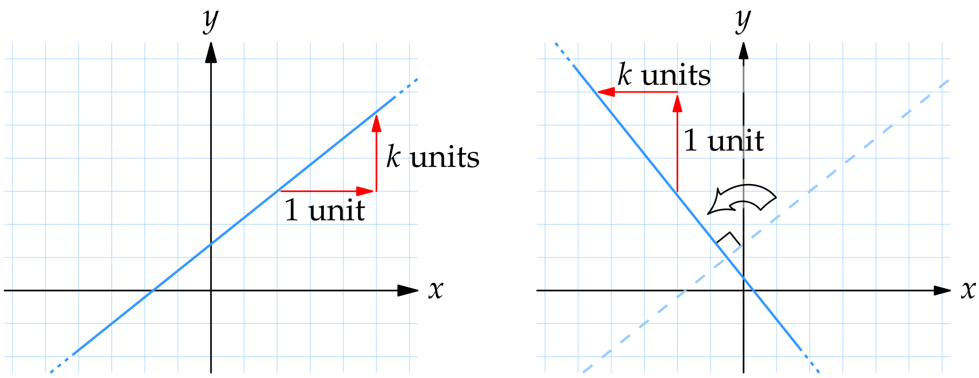
Line $y = 2 - x/2$

Example 7

What is the gradient of the straight line that passes through the points $(2, 1)$ and $(5, 3)$?

If we plot the points and draw the line in a coordinate system, we see that $5 - 2 = 3$ steps in the x-direction correspond to $3 - 1 = 2$ steps in the y-direction along the line. This means that 1 step in the x-direction corresponds to $m = \frac{3-1}{5-2} = \frac{2}{3}$ steps in the y-direction. So the line's gradient is $m = \frac{2}{3}$.

Two straight lines that are parallel clearly have the same gradient. It is also possible to see (such as in the figure below) that if two lines with gradients m_1 and m_2 are perpendicular to one another, then $m_2 = -\frac{1}{m_1}$, which also can be written as $m_1 m_2 = -1$.



The straight line in the figure on the left has gradient m, that is 1 step in the x-direction corresponds to m steps in the y-direction. If the line is rotated 90° clockwise we get the line in the figure to the right. That line has gradient $-\frac{1}{m}$ because now $-m$ steps in the x-direction corresponds to 1 step in the y-direction.

Example 8

a) The lines $y = 3x - 1$ and $y = 3x + 5$ are parallel.

b) The lines $y = x + 1$ and $y = 2 - x$ are perpendicular.

All straight lines (including vertical lines) can be put into the general form

$$ax + by = c$$

where a, b and c are constants.

Example 9

a) Put the line $y = 5x + 7$ into the form $ax + by = c$.

Move the x-term to the left-hand side: $-5x + y = 7$.

b) Put the line $2x + 3y = -1$ into the form $y = mx + c$.

Move the x-term to the right-hand side

$$3y = -2x - 1$$

and divide both sides by 3

$$y = -\tfrac{2}{3}x - \tfrac{1}{3}.$$

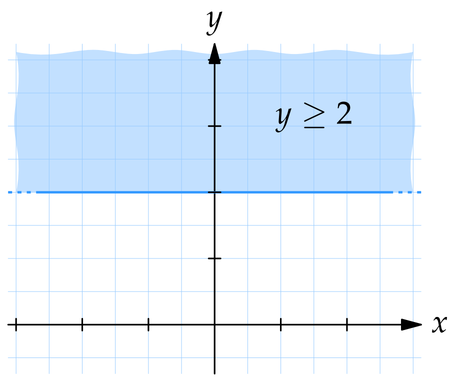
Regions in a coordinate system

By geometrically interpreting inequalities one can describe regions in the plane.

Example 10

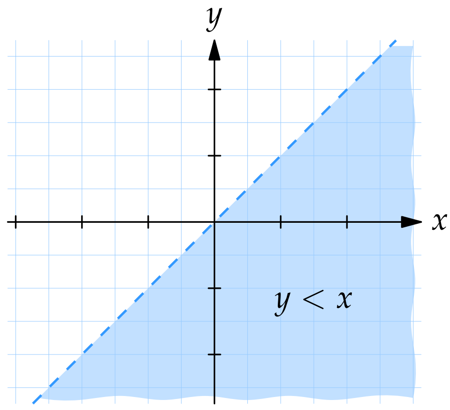
a) Sketch the region in the x, y-plane that satisfies $y \geq 2$.

The region is given by all the points (x, y) for which the y-coordinate is equal to, or greater than, 2 that is all points on or above the line $y = 2$.



b) Sketch the region in the x, y-plane that satisfies $y < x$.

A point (x, y) that satisfies the inequality $y < x$ must have an x-coordinate that is larger than its y-coordinate. Thus the area consists of all the points to the right of the line $y = x$. The fact that the line $y = x$ is shown dashed signifies that the points on the line do not belong to the shaded area.



Example 11

Sketch the region in the x, y-plane that satisfies $2 \leq 3x + 2y \leq 4$.

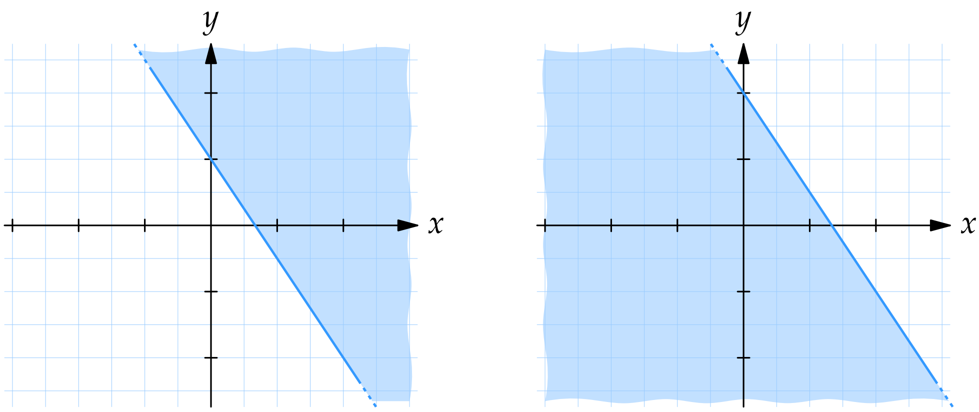
The double inequality can be divided into two inequalities

$$3x + 2y \geq 2 \quad \text{und} \quad 3x + 2y \leq 4.$$

We move the x-terms into the right-hand side and divide both sides by 2 giving

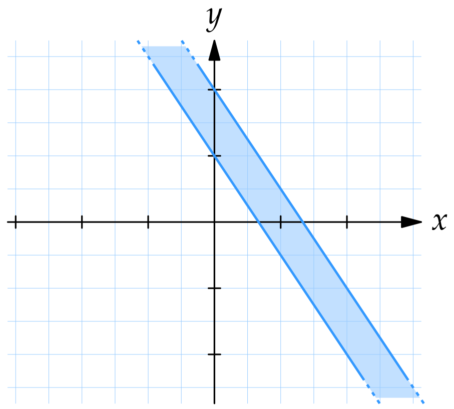
$$y \geq 1 - \frac{3}{2}x \quad \text{und} \quad y \leq 2 - \frac{3}{2}x.$$

The points that satisfy the first inequality are on and above the line $y = 1 - \frac{3}{2}x$ while the points that satisfy the other inequality are on or below the line $y = 2 - \frac{3}{2}x$.



The figure on the left shows the region $3x + 2y \geq 2$ and figure to the right shows the region $3x + 2y \leq 4$.

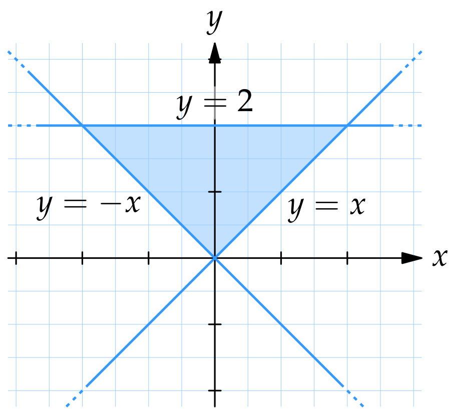
Points that satisfy both inequalities form a band-like region where both shaded areas overlap.



The figure shows the region $2 \leq 3x + 2y \leq 4$.

Example 12

If we draw the lines $y = x$, $y = -x$ and $y = 2$ then these lines bound a triangle in the xy plane.



We find that for a point to lie in this triangle, it has to satisfy certain conditions.

We see that its y-coordinate must be less than 2. At the same time, we see that the triangle is bounded by $y = 0$ below. Thus the y coordinates must be in the range $0 \leq y \leq 2$.

For the x-coordinate the situation is a little more complicated. We see that the x-coordinate must satisfy the fact that all points lie above the lines $y = -x$ and $y = x$. We see that this is satisfied if $-y \leq x \leq y$. Since we already have restricted the y-coordinates we find that x cannot be larger than 2 or less than -2.

This gives the base of the triangle as 4 units of length and a height of 2 units of length.

The area of this triangle is therefore $4 \cdot 2/2 = 4$ square units.

Study advice

Basic and final tests
After you have read the text and worked through the exercises you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that . . .
You should draw your own diagrams when you solve geometrical problems and draw them carefully and accurately! A good diagram can mean you are halfway to a solution, however a poor diagram may well fool you.

Useful web sites

- Experiment with Equations of a Straight Line
  (`http://www.cut-the-knot.org/Curriculum/Calculus/StraightLine.shtml`)
- Experiment with Archimedes Triangle and Squaring of Parabola
  (`http://www.cut-the-knot.org/Curriculum/Geometry/ArchimedesTriangle.shtml`)

2.2 Exercises

Exercise 2.2:1

Solve the equations

a) $x - 2 = -1$

b) $2x + 1 = 13$

c) $\frac{1}{3}x - 1 = x$

d) $5x + 7 = 2x - 6$

Exercise 2.2:2

Solve the equations

a) $\dfrac{5x}{6} - \dfrac{x+2}{9} = \dfrac{1}{2}$

b) $\dfrac{8x+3}{7} - \dfrac{5x-7}{4} = 2$

c) $(x+3)^2 - (x-5)^2 = 6x + 4$

d) $(x^2 + 4x + 1)^2 + 3x^4 - 2x^2 = (2x^2 + 2x + 3)^2$

Exercise 2.2:3

Solve the equations

a) $\dfrac{x+3}{x-3} - \dfrac{x+5}{x-2} = 0$

b) $\dfrac{4x}{4x-7} - \dfrac{1}{2x-3} = 1$

c) $\left(\dfrac{1}{x-1} - \dfrac{1}{x+1}\right)\left(x^2 + \frac{1}{2}\right) = \dfrac{6x-1}{3x-3}$

d) $\left(\dfrac{2}{x} - 3\right)\left(\dfrac{1}{4x} + \dfrac{1}{2}\right) - \left(\dfrac{1}{2x} - \dfrac{2}{3}\right)^2 - \left(\dfrac{1}{2x} + \dfrac{1}{3}\right)\left(\dfrac{1}{2x} - \dfrac{1}{3}\right) = 0$

Exercise 2.2:4

a) Write the equation for the line $y = 2x + 3$ in the form $ax + by = c$.

b) Write the equation for the line $3x + 4y - 5 = 0$ in the form $y = kx + m$.

Exercise 2.2:5

a) Determine the equation for the straight line that goes between the points $(2,3)$ and $(3,0)$.

b) Determine the equation for the straight line that has slope -3 and goes through the point $(1,-2)$.

c) Determine the equation for the straight line that goes through the point $(-1,2)$ and is parallel to the line $y = 3x + 1$.

d) Determine the equation for the straight line that goes through the point $(2,4)$ and is perpendicular to the line $y = 2x + 5$.

e) Determine the slope, k, for the straight line that cuts the x-axis at the point $(5,0)$ and y-axis at the point $(0,-8)$.

Exercise 2.2:6

Find the points of intersection between the pairs of lines in the following

a) $y = 3x + 5$ and the x-axis

b) $y = -x + 5$ and the y-axis

c) $4x + 5y + 6 = 0$ and the y-axis

d) $x + y + 1 = 0$ and $x = 12$

e) $2x + y - 1 = 0$ and $y - 2x - 2 = 0$

Exercise 2.2:7

Sketch the graph of the functions

a) $f(x) = 3x - 2$

b) $f(x) = 2 - x$

c) $f(x) = 2$

Exercise 2.2:8

In the xy-plane, shade in the section whose coordinates (x, y) satisfy

a) $y \geq x$

b) $y < 3x - 4$

c) $2x + 3y \leq 6$

Exercise 2.2:9

Calculate the area of the triangle which

a) has corners at the points $(1, 4)$, $(3, 3)$ and $(1, 0)$.

b) is bordered by the lines $x = 2y$, $y = 4$ and $y = 10 - 2x$.

c) is described by the inequalities $x + y \geq -2$, $2x - y \leq 2$ and $2y - x \leq 2$.

2.3 Quadratic expressions

Contents:

- Completing the square
- Quadratic equations
- Factorising
- Parabolas

Learning outcomes:

After this section, you will have learned to:

- Complete the square in quadratic expressions.
- Solve quadratic equations by completing the square (not using a standard formula) and know how to check the answer.
- Factorise quadratic expressions (when possible).
- Directly solve factorised or almost factorised quadratic equations.
- Determine the minimum / maximum value of a quadratic expression.
- Sketch parabolas by completing the square.

Quadratic equations

A quadratic equation is one that can be written as

$$x^2 + px + q = 0$$

where x is the unknown and p and q are constants.

The simplest forms of quadratic equations can be solved directly by taking roots.

The equation $x^2 = a$ where a is a positive number has two solutions (roots) $x = \sqrt{a}$ and $x = -\sqrt{a}$.

Example 1

a) $x^2 = 4$ has the roots $x = \sqrt{4} = 2$ and $x = -\sqrt{4} = -2$.

b) $2x^2 = 18$ is rewritten as $x^2 = 9$ and has the roots $x = \sqrt{9} = 3$ and $x = -\sqrt{9} = -3$.

c) $3x^2 - 15 = 0$ can be rewritten as $x^2 = 5$ and has the roots $x = \sqrt{5} \approx 2.236$ and $x = -\sqrt{5} \approx -2.236$.

d) $9x^2 + 25 = 0$ has no solutions because the left-hand side will always be greater than or equal to 25 regardless of the value of x (being a square, x^2 is always greater than or equal to zero).

Example 2

a) Solve the equation $(x-1)^2 = 16$.

By considering $x - 1$ as the unknown and taking the roots one finds the equation has two solutions

- $x - 1 = \sqrt{16} = 4$ which gives $x = 1 + 4 = 5$.
- $x - 1 = -\sqrt{16} = -4$ which gives $x = 1 - 4 = -3$.

b) Solve the equation $2(x+1)^2 - 8 = 0$.

Move the term 8 over to the right-hand side and divide both sides by 2,

$$(x+1)^2 = 4.$$

Taking the roots gives:

- $x + 1 = \sqrt{4} = 2$, which gives $x = -1 + 2 = 1$,
- $x + 1 = -\sqrt{4} = -2$, which gives $x = -1 - 2 = -3$.

To solve a quadratic equation more generally we can use a technique called completing the square.

If we consider the rule for expanding a quadratic,

$$x^2 + 2ax + a^2 = (x+a)^2$$

and subtract the a^2 from both sides we get

Completing the square:

$$x^2 + 2ax = (x + a)^2 - a^2.$$

Exempel 3

a) Solve the equation $x^2 + 2x - 8 = 0$.

One completes the square for $x^2 + 2x$ (use $a = 1$ in the formula)

$$\underline{x^2 + 2x} - 8 = \underline{(x + 1)^2 - 1^2} - 8 = (x + 1)^2 - 9,$$

where the underlined terms are those involved in the completion of the square. Using this we can write

$$(x + 1)^2 - 9 = 0,$$

which we solve by taking roots, namely

- $x + 1 = \sqrt{9} = 3\,,\quad$ giving $\quad x = -1 + 3 = 2,$
- $x + 1 = -\sqrt{9} = -3\,,\quad$ giving $\quad x = -1 - 3 = -4.$

b) Solve the equation $2x^2 - 2x - \frac{3}{2} = 0$.

Divide both sides by 2

$$x^2 - x - \tfrac{3}{4} = 0.$$

Complete the square on the left-hand side (use $a = -\frac{1}{2}$),

$$\underline{x^2 - x} - \tfrac{3}{4} = \underline{\bigl(x - \tfrac{1}{2}\bigr)^2 - \bigl(-\tfrac{1}{2}\bigr)^2} - \tfrac{3}{4} = \bigl(x - \tfrac{1}{2}\bigr)^2 - 1.$$

This gives us the equation

$$\bigl(x - \tfrac{1}{2}\bigr)^2 - 1 = 0.$$

Taking roots gives

- $x - \frac{1}{2} = \sqrt{1} = 1,\quad$ i.e. $\quad x = \frac{1}{2} + 1 = \frac{3}{2},$
- $x - \frac{1}{2} = -\sqrt{1} = -1,\quad$ i.e. $\quad x = \frac{1}{2} - 1 = -\frac{1}{2}.$

Hint:

Keep in mind that we can always test our solution of an equation by inserting the solution's value into the equation and checking that it is satisfied. We should always do this to check for any careless mistakes. For example, in 3a above we have two cases to consider. We call the left-hand and right-hand sides LHS and RHS respectively.

- $x = 2$ gives that $\text{LHS} = 2^2 + 2 \cdot 2 - 8 = 4 + 4 - 8 = 0 = \text{RHS}$.
- $x = -4$ gives that $\text{LHS} = (-4)^2 + 2 \cdot (-4) - 8 = 16 - 8 - 8 = 0 = \text{RHS}$.

In both cases we arrive at LHS = RHS. The equation is satisfied in both cases.

Using the completing the square method it is possible to show that the general quadratic equation

$$x^2 + px + q = 0$$

has the solutions

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q},$$

provided that the term inside the root sign is not negative.

Sometimes one can factorise the equations directly and thus immediately see what the solutions are.

Example 4

a) Solve the equation $x^2 - 4x = 0$.

On the left-hand side we can factor out an x

$$x(x-4) = 0.$$

The equation on the lefthand side is zero when one of its factors is zero, which gives us two solutions

- $x = 0$, or
- $x - 4 = 0$ which gives $x = 4$.

Parabolas

The functions

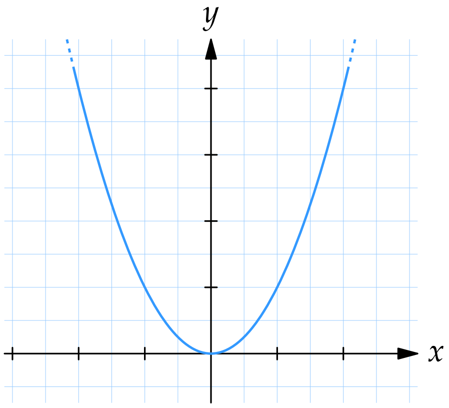
$$y = x^2 - 2x + 5,$$
$$y = 4 - 3x^2 \quad \text{and}$$
$$y = \tfrac{1}{5}x^2 + 3x$$

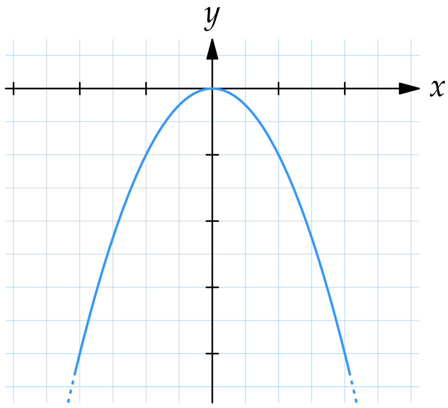
are examples of quadratic functions. In general a quadratic function can be written as

$$y = ax^2 + bx + c,$$

where a, b, c are constants and $a \neq 0$.

The graph for a quadratic function is known as a parabola. The figures show the graphs of two typical parabolas $y = x^2$ and $y = -x^2$.





The figure on the left shows the parabola $y = x^2$. The figure to the right is the parabola $y = -x^2$.

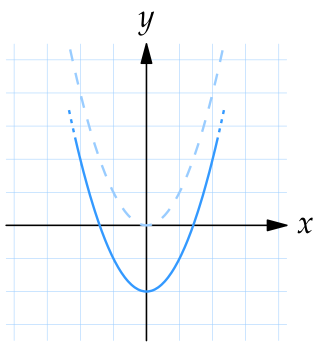
As the expression x^2 is minimal when $x = 0$, the parabola $y = x^2$ has a minimum when $x = 0$. Similarly, the parabola $y = -x^2$ has a maximum when $x = 0$.

Note also that the parabolas above are symmetrical about the y-axis. This is because the value of x^2 does not depend on the sign of x.

Example 5

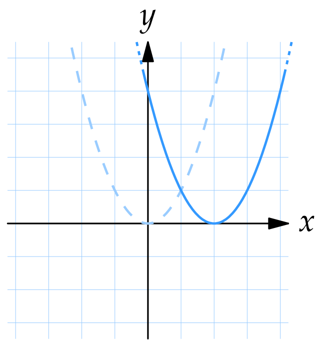
a) Sketch the parabola $y = x^2 - 2$.

Comparing it to the parabola $y = x^2$, we see that all points on the parabola ($y = x^2 - 2$) will have y-values that are two units less, so the parabola has been displaced downwards two units along the y-direction.



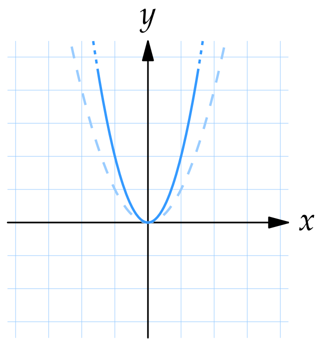
b) Sketch the parabola $y = (x-2)^2$.

For the parabola $y = (x-2)^2$, we need to choose x-values that are two units larger than those for parabola $y = x^2$ to get the same y value. So the parabola $y = (x-2)^2$ has been displaced two units to the right.



c) Sketch the parabola $y = 2x^2$.

Each point on the parabola $y = 2x^2$ has a y-value twice as large as the point with the same x-value on the parabola $y = x^2$. Thus, the parabola $y = 2x^2$ has been stretched by a factor of 2 in the y-direction in comparison to $y = x^2$.



All sorts of parabolas can be handled by completing the square.

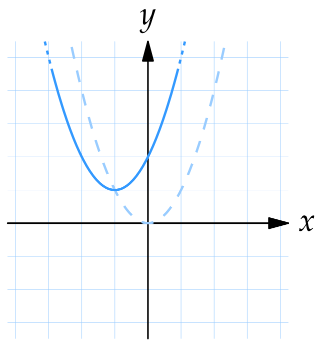
Example 6

Sketch the parabola $y = x^2 + 2x + 2$.

If one completes the square for the right-hand side

$$x^2 + 2x + 2 = (x+1)^2 - 1^2 + 2 = (x+1)^2 + 1$$

we see from the resulting expression $y = (x+1)^2 + 1$ that the parabola has been displaced one unit to the left along the x-direction and one unit up in the y-direction, as compared to $y = x^2$.



Example 7

Determine where the parabola $y = x^2 - 4x + 3$ intersects the x-axis.

A point is on the x-axis if its y-coordinate is zero. The points on the parabola which have $y = 0$ have an x-coordinate that satisfies the equation

$$x^2 - 4x + 3 = 0.$$

Complete the square for the left-hand side

$$x^2 - 4x + 3 = (x-2)^2 - 2^2 + 3 = (x-2)^2 - 1$$

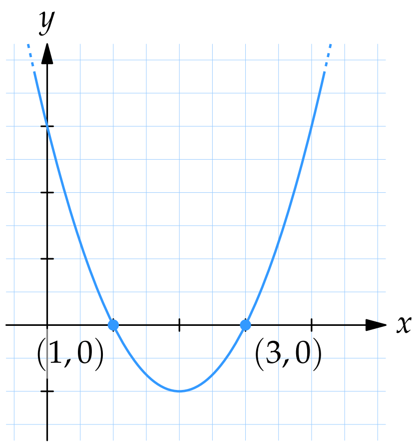
and this gives the equation

$$(x-2)^2 = 1.$$

After taking roots we get the solutions

- $x - 2 = \sqrt{1} = 1, \quad$ i.e. $\quad x = 2 + 1 = 3,$
- $x - 2 = -\sqrt{1} = -1, \quad$ i.e. $\quad x = 2 - 1 = 1.$

The parabola cuts the x-axis in points $(1,0)$ and $(3,0)$.



Example 8

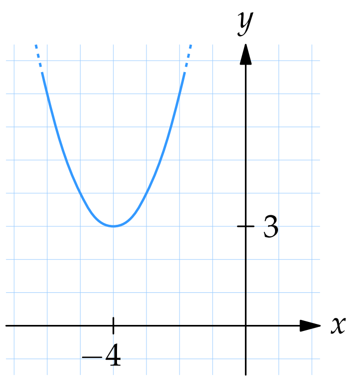
Determine the minimum value of the expression $x^2 + 8x + 19$.

We complete the square

$$x^2 + 8x + 19 = (x+4)^2 - 4^2 + 19 = (x+4)^2 + 3$$

and see that the y-value must always be greater than or equal to 3. This is because the square $(x+4)^2$ is always greater than or equal to 0 regardless of what x is.

In the figure below we see that the whole parabola $y = x^2 + 8x + 19$ lies above the x-axis and has a minimum 3 at $x = -4$.



Study advice

Basic and final tests
After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that...
You should devote a lot of time to doing algebra! Algebra is the alphabet of mathematics. Once you understand algebra you will enhance your understanding of statistics, areas, volumes and geometry.

Reviews
For those of you who want to deepen your studies or need more detailed explanations consider the following references

- Learn more about quadratic equations in the English Wikipedia (`http://en.wikipedia.org/wiki/Quadratic_equation`)
- Learn more about quadratic equations in mathworld (`http://mathworld.wolfram.com/QuadraticEquation.html`)
- 101 uses of a quadratic equation — by Chris Budd and Chris Sangwin (`http://plus.maths.org/issue29/features/quadratic/index-gifd.html`)

2.3 Exercises

Exercise 2.3:1

Complete the square of the expressions

a) $x^2 - 2x$ b) $x^2 + 2x - 1$ c) $5 + 2x - x^2$ d) $x^2 + 5x + 3$

Exercise 2.3:2

Solve the following second order equations by completing the square

a) $x^2 - 4x + 3 = 0$ b) $y^2 + 2y - 15 = 0$ c) $y^2 + 3y + 4 = 0$

d) $4x^2 - 28x + 13 = 0$ e) $5x^2 + 2x - 3 = 0$ f) $3x^2 - 10x + 8 = 0$

Exercise 2.3:3

Solve the following equations directly

a) $x(x+3) = 0$ b) $(x-3)(x+5) = 0$

c) $5(3x-2)(x+8) = 0$ d) $x(x+3) - x(2x-9) = 0$

e) $(x+3)(x-1) - (x+3)(2x-9) = 0$ f) $x(x^2 - 2x) + x(2-x) = 0$

Exercise 2.3:4

Find a second-degree equation which has roots

a) -1 and 2

b) $1+\sqrt{3}$ and $1-\sqrt{3}$

c) 3 and $\sqrt{3}$

Exercise 2.3:5

a) Find a second-degree equation which only has -7 as a root.

b) Determine a value of x which makes the expression $4x^2 - 28x + 48$ negative.

c) The equation $x^2 + 4x + b = 0$ has one root at $x = 1$. Determine the value of the constant b.

Exercise 2.3:6

Determine the smallest value that the following polynomials can take

a) $x^2 - 2x + 1$ b) $x^2 - 4x + 2$ c) $x^2 - 5x + 7$

Exercise 2.3:7

Determine the largest value that the following polynomials can take

a) $1 - x^2$ b) $-x^2 + 3x - 4$ c) $x^2 + x + 1$

Exercise 2.3:8

Sketch the graph of the following functions

a) $f(x) = x^2 + 1$ b) $f(x) = (x-1)^2 + 2$ c) $f(x) = x^2 - 6x + 11$

Exercise 2.3:9

Find all the points where the following curves intersect the x-axis.

a) $y = x^2 - 1$ b) $y = x^2 - 5x + 6$ c) $y = 3x^2 - 12x + 9$

Exercise 2.3:10

In the xy-plane, shade in the area whose coordinates (x, y) satisfy

a) $y \geq x^2$ and $y \leq 1$ b) $y \leq 1 - x^2$ and $x \geq 2y - 3$

c) $1 \geq x \geq y^2$ d) $x^2 \leq y \leq x$

3.1 Roots

Contents:

- Square roots and nth roots
- Manipulating roots

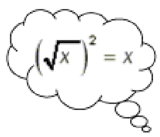
Learning outcomes:

After this section, you will have learned:

- How to calculate the square root of some simple integers.
- That the square root of a negative number is not defined.
- That the square root of a number denotes the positive root.
- How to manipulate roots in the simplification of expressions.
- To recognise when the methods of manipulating roots are valid.
- How to simplify expressions containing square roots in the denominator.
- When the nth root of a negative number is defined (n odd).

Square roots

The well-known symbol $\sqrt{a}$, the square root of a, is used to describe the number that when multiplied by itself gives a. However, one has to be a little more precise in defining this symbol.



The equation $x^2 = 4$ has two solutions $x = 2$ and $x = -2$, since both $2 \cdot 2 = 4$ and $(-2) \cdot (-2) = 4$. It would then be logical to suppose that $\sqrt{4}$ can be either -2 or 2, i.e. $\sqrt{4} = \pm 2$, but by convention, $\sqrt{4}$ only denotes the positive number 2.

The square root $\sqrt{a}$ means **the non-negative number** that, when multiplied by itself, gives a; that is, the non-negative solution of the equation $x^2 = a$.

The square root of a can also be written as $a^{1/2}$.

It is therefore wrong to state that $\sqrt{4} = \pm 2$, but correct to state that the equation $x^2 = 4$ has the solution $x = \pm 2$.

Example 1

a) $\sqrt{0} = 0$ because $0^2 = 0 \cdot 0 = 0$ and 0 is not negative.

b) $\sqrt{100} = 10$ since $10^2 = 10 \cdot 10 = 100$ and 10 is a positive number.

c) $\sqrt{0.25} = 0.5$ since $0.5^2 = 0.5 \cdot 0.5 = 0.25$ and 0.5 is positive.

d) $\sqrt{2} \approx 1.4142$ since $1.4142 \cdot 1.4142 \approx 2$ and 1.4142 is positive.

e) The equation $x^2 = 2$ has the solutions $x = \sqrt{2} \approx 1.4142$ and $x = -\sqrt{2} \approx -1.4142$.

f) $\sqrt{-4}$ is not defined, since there is no real number x that satisfies $x^2 = -4$.

g) $\sqrt{(-7)^2} = 7$ because $\sqrt{(-7)^2} = \sqrt{(-7) \cdot (-7)} = \sqrt{49} = \sqrt{7 \cdot 7} = 7$.

It is useful to know how square roots behave in calculations. As $\sqrt{a} = a^{1/2}$, we can handle expressions involving roots as we would expressions involving exponents. For example, we have

$$\sqrt{9 \cdot 4} = (9 \cdot 4)^{1/2} = 9^{1/2} \cdot 4^{1/2} = \sqrt{9} \cdot \sqrt{4}.$$

In this way we obtain the following rules for square roots.

For all real numbers $a, b \geq 0$:

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$a\sqrt{b} = \sqrt{a^2 b}$$

(In the division the number b must, of course, be non-zero.)

Example 2

a) $\sqrt{64 \cdot 81} = \sqrt{64} \cdot \sqrt{81} = 8 \cdot 9 = 72$

b) $\sqrt{\dfrac{9}{25}} = \dfrac{\sqrt{9}}{\sqrt{25}} = \dfrac{3}{5}$

c) $\sqrt{18} \cdot \sqrt{2} = \sqrt{18 \cdot 2} = \sqrt{36} = 6$

d) $\dfrac{\sqrt{75}}{\sqrt{3}} = \sqrt{\dfrac{75}{3}} = \sqrt{25} = 5$

e) $\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$

Note that the above calculations assume that a and $b \geq 0$. If a and b are negative (< 0) then $\sqrt{a}$ and $\sqrt{b}$ are not defined as real numbers. It is tempting to write, for example,

$$-1 = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1) \cdot (-1)} = \sqrt{1} = 1$$

but something here cannot be right. The explanation is that $\sqrt{-1}$ is not a real number, which means the laws of roots discussed above may not be used.

Higher order roots

The cube root of a number a is defined as the number that multiplied by itself three times gives a, and is denoted as $\sqrt[3]{a}$.

Example 3

a) $\sqrt[3]{8} = 2 \quad$ as $2 \cdot 2 \cdot 2 = 8$.

b) $\sqrt[3]{0{,}027} = 0{,}3 \quad$ since $0{,}3 \cdot 0{,}3 \cdot 0{,}3 = 0{,}027$.

c) $\sqrt[3]{-8} = -2 \quad$ because $(-2) \cdot (-2) \cdot (-2) = -8$.

Note that, unlike square roots, cube roots are also defined for negative numbers.

For any positive integer n one can define the nth root of a number a:

- if n is even and $a \geq 0$ then $\sqrt[n]{a}$ is the non-negative number that when multiplied by itself n times gives a,
- if n is odd, $\sqrt[n]{a}$ is the number that when multiplied by itself n times gives a.

The root $\sqrt[n]{a}$ can also be written as $a^{1/n}$.

Example 4

a) $\sqrt[4]{625} = 5$ since $5 \cdot 5 \cdot 5 \cdot 5 = 625$.

b) $\sqrt[5]{-243} = -3$ because $(-3) \cdot (-3) \cdot (-3) \cdot (-3) \cdot (-3) = -243$.

c) $\sqrt[6]{-17}$ is not defined, since 6 is even and -17 is a negative number.

For nth roots the same rules apply as for square roots if $a,\ b \geq 0$. Note that if n is odd these methods apply even for negative a and b, that is, for all real numbers a and b.

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$a\sqrt[n]{b} = \sqrt[n]{a^n b}$$

(In the division the number b must, of course, be non-zero.)

Surds

Some square roots, such as $\sqrt{9}$, can be expressed as exact numbers. Others, like $\sqrt{2}$, cannot, because they are irrational and therefore their decimal forms go on forever without any repeating patterns.

Of course, numbers like $\sqrt{2}$ can be expressed "approximately" as decimals, but in fact it is often preferable to leave expressions like $\sqrt{2}$ unevaluated, at least until the end of a calculation; that way, there is no loss of accuracy along the way.

Expressions containing unevaluated irrational square roots, or more generally nth roots, are often called surds.

Simplification of expressions containing roots

Often one can significantly simplify surds — expressions containing roots — by using the rules described above. As with indices, it is desirable to reduce expressions into "small" roots. For example, it is a good idea to do the following

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2},$$

because it may help with later simplification, as we see here

$$\frac{\sqrt{8}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}.$$

By rewriting surds in terms of "small" roots one can also sum roots of "the same kind", e.g.

$$\sqrt{8} + \sqrt{2} = 2\sqrt{2} + \sqrt{2} = (2+1)\sqrt{2} = 3\sqrt{2}.$$

Example 5

a) $\dfrac{\sqrt{8}}{\sqrt{18}} = \dfrac{\sqrt{2\cdot 4}}{\sqrt{2\cdot 9}} = \dfrac{\sqrt{2\cdot 2\cdot 2}}{\sqrt{2\cdot 3\cdot 3}} = \dfrac{\sqrt{2\cdot 2^2}}{\sqrt{2\cdot 3^2}} = \dfrac{2\sqrt{2}}{3\sqrt{2}} = \dfrac{2}{3}$

b) $\dfrac{\sqrt{72}}{6} = \dfrac{\sqrt{8\cdot 9}}{2\cdot 3} = \dfrac{\sqrt{2\cdot 2\cdot 2\cdot 3\cdot 3}}{2\cdot 3} = \dfrac{\sqrt{2^2\cdot 3^2\cdot 2}}{2\cdot 3} = \dfrac{2\cdot 3\sqrt{2}}{2\cdot 3} = \sqrt{2}$

c) $$\begin{aligned}\sqrt{45} + \sqrt{20} &= \sqrt{9\cdot 5} + \sqrt{4\cdot 5} = \sqrt{3^2\cdot 5} + \sqrt{2^2\cdot 5} = 3\sqrt{5} + 2\sqrt{5}\\ &= (3+2)\sqrt{5} = 5\sqrt{5}\end{aligned}$$

d) $$\begin{aligned}\sqrt{50} + 2\sqrt{3} - \sqrt{32} + \sqrt{27} &= \sqrt{5\cdot 10} + 2\sqrt{3} - \sqrt{2\cdot 16} + \sqrt{3\cdot 9}\\ &= \sqrt{5\cdot 2\cdot 5} + 2\sqrt{3} - \sqrt{2\cdot 4\cdot 4} + \sqrt{3\cdot 3\cdot 3}\\ &= \sqrt{5^2\cdot 2} + 2\sqrt{3} - \sqrt{2^2\cdot 2^2\cdot 2} + \sqrt{3\cdot 3^2}\\ &= 5\sqrt{2} + 2\sqrt{3} - 2\cdot 2\sqrt{2} + 3\sqrt{3}\\ &= (5-4)\sqrt{2} + (2+3)\sqrt{3}\\ &= \sqrt{2} + 5\sqrt{3}\end{aligned}$$

e) $$\begin{aligned}\frac{2\cdot\sqrt[3]{3}}{\sqrt[3]{12}} &= \frac{2\cdot\sqrt[3]{3}}{\sqrt[3]{3\cdot 4}} = \frac{2\cdot\sqrt[3]{3}}{\sqrt[3]{3}\cdot\sqrt[3]{4}} = \frac{2}{\sqrt[3]{4}} = \frac{2}{\sqrt[3]{2\cdot 2}}\\ &= \frac{2}{\sqrt[3]{2}\cdot\sqrt[3]{2}}\cdot\frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{2\cdot\sqrt[3]{2}}{2} = \sqrt[3]{2}\end{aligned}$$

f) $(\sqrt{3} + \sqrt{2}\,)(\sqrt{3} - \sqrt{2}\,) = (\sqrt{3}\,)^2 - (\sqrt{2}\,)^2 = 3 - 2 = 1$

where we have used the difference of two squares $(a+b)(a-b) = a^2 - b^2$ with $a = \sqrt{3}$ and $b = \sqrt{2}$.

Rationalising the denominator

When roots appear in a rational expression it is often useful to write the expression in a form which does not contain any roots in the denominator. This is because it is difficult to divide by irrational numbers by hand. In the example below, multiplying by $1 = \frac{\sqrt{2}}{\sqrt{2}}$, one obtains

$$\frac{1}{\sqrt{2}} = \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{2}}{2}.$$

which is usually preferable. This is called rationalising the denominator.

In slightly more complicated cases, we can use the difference of two squares, $(a + b)(a - b) = a^2 - b^2$, to eliminate the root from the denominator. The trick is this: if the denominator is of the form $a + b$, where either a or b (or both) contains a square root, then we multiply the numerator and denominator by $a - b$; if the denominator is of the form $a - b$, we multiply by $a + b$. For example,

$$\begin{aligned} \frac{\sqrt{3}}{\sqrt{2}+1} &= \frac{\sqrt{3}}{\sqrt{2}+1} \cdot \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{\sqrt{3}\,(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} \\ &= \frac{\sqrt{3} \cdot \sqrt{2} - \sqrt{3} \cdot 1}{(\sqrt{2}\,)^2 - 1^2} = \frac{\sqrt{3 \cdot 2} - \sqrt{3}}{2-1} = \frac{\sqrt{6}-\sqrt{3}}{1} = \sqrt{6} - \sqrt{3}. \end{aligned}$$

Example 6

a) $$\frac{10\sqrt{3}}{\sqrt{5}} = \frac{10\sqrt{3} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \frac{10\sqrt{15}}{5} = 2\sqrt{15}$$

b) $$\frac{1+\sqrt{3}}{\sqrt{2}} = \frac{(1+\sqrt{3}) \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{2}$$

c) $$\frac{3}{\sqrt{2}-2} = \frac{3(\sqrt{2}+2)}{(\sqrt{2}-2)(\sqrt{2}+2)} = \frac{3\sqrt{2}+6}{(\sqrt{2}\,)^2 - 2^2} = \frac{3\sqrt{2}+6}{2-4} = -\frac{3\sqrt{2}+6}{2}$$

d) $$\begin{aligned} \frac{\sqrt{2}}{\sqrt{6}+\sqrt{3}} &= \frac{\sqrt{2}\,(\sqrt{6}-\sqrt{3}\,)}{(\sqrt{6}+\sqrt{3}\,)(\sqrt{6}-\sqrt{3}\,)} = \frac{\sqrt{2}\,\sqrt{6} - \sqrt{2}\,\sqrt{3}}{(\sqrt{6}\,)^2 - (\sqrt{3}\,)^2} \\ &= \frac{\sqrt{2}\,\sqrt{2 \cdot 3} - \sqrt{2}\,\sqrt{3}}{6-3} = \frac{2\sqrt{3} - \sqrt{2}\,\sqrt{3}}{3} = \frac{(2-\sqrt{2}\,)\sqrt{3}}{3} \end{aligned}$$

Study advice

Basic and final tests
After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that . . .
The square root of a number is always non-negative (that is, positive or zero)!

Rules for roots are actually a special case of laws of exponents, since, for example, $\sqrt{x} = x^{1/2}$.

Reviews
For those of you who want to deepen your understanding or need more detailed explanations consider the following references

- Learn more about square roots from Wikipedia
(`http://en.wikipedia.org/wiki/Root_(mathematics)`)
- How do we know that the square root of 2 is not a fraction?
(`http://www.mathacademy.com/pr/prime/articles/irr2/`)

Useful web sites

- How to find the root of a number, without the help of calculators?
(`http://mathforum.org/dr.math/faq/faq.sqrt.by.hand.html`)

3.1 Exercises

Exercise 3.1:1

Write in power form

a) $\sqrt{2}$ b) $\sqrt{7^5}$ c) $\left(\sqrt[3]{3}\right)^4$ d) $\sqrt{\sqrt{3}}$

Exercise 3.1:2

Write in simplest possible form.

a) $\sqrt{3^2}$ b) $\sqrt{(-3)^2}$ c) $\sqrt{-3^2}$ d) $\sqrt{5}\cdot\sqrt[3]{5}\cdot 5$

e) $\sqrt{18}\cdot\sqrt{8}$ f) $\sqrt[3]{8}$ g) $\sqrt[3]{-125}$

Exercise 3.1:3

Write in simplest possible form.

a) $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)$ b) $\dfrac{\sqrt{96}}{\sqrt{18}}$

c) $\sqrt{16+\sqrt{16}}$ d) $\sqrt{\dfrac{2}{3}}\left(\sqrt{6}-\sqrt{3}\right)$

Exercise 3.1:4

Write in simplest possible form.

a) $\sqrt{0{,}16}$ b) $\sqrt[3]{0{,}027}$

c) $\sqrt{50}+4\sqrt{20}-3\sqrt{18}-2\sqrt{80}$ d) $\sqrt{48}+\sqrt{12}+\sqrt{3}-\sqrt{75}$

Exercise 3.1:5

Write as an expression without a root sign in the denominator.

a) $\dfrac{2}{\sqrt{12}}$ b) $\dfrac{1}{\sqrt[3]{7}}$ c) $\dfrac{2}{3+\sqrt{7}}$ d) $\dfrac{1}{\sqrt{17}-\sqrt{13}}$

Exercise 3.1:6

Write as an expression without a root sign in the denominator.

a) $\dfrac{\sqrt{2}+3}{\sqrt{5}-2}$ b) $\dfrac{1}{(\sqrt{3}-2)^2-2}$

c) $\dfrac{\dfrac{1}{\sqrt{3}}-\dfrac{1}{\sqrt{5}}}{\dfrac{1}{\sqrt{2}}-\dfrac{1}{2}}$ d) $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{6}}$

Exercise 3.1:7

Write in simplest possible form.

a) $\dfrac{1}{\sqrt{6}-\sqrt{5}} - \dfrac{1}{\sqrt{7}-\sqrt{6}}$

b) $\dfrac{5\sqrt{7}-7\sqrt{5}}{\sqrt{7}-\sqrt{5}}$

c) $\sqrt{153}-\sqrt{68}$

Exercise 3.1:8

Determine which number is the larger:

a) $\sqrt[3]{5}$ and $\sqrt[3]{6}$

b) $\sqrt{7}$ and 7

c) $\sqrt{7}$ and $2{,}5$

d) $\sqrt{2}\left(\sqrt[4]{3}\right)^3$ and $\sqrt[3]{2}\cdot 3$

3.2 Equations with roots

Contents:

- Equations of the type $\sqrt{ax+b} = cx+d$
- Spurious roots

Learning outcomes:

After this section, you will have learned to:

- Solve simple equations containing roots.
- Manage spurious roots, and know when they might appear.

Equations with roots

There are many different types of equations containing roots. Some examples are

$$\sqrt{x} + 3x = 2,$$
$$\sqrt{x-1} - 2x = x^2,$$
$$\sqrt[3]{x+2} = x.$$

To solve equations with roots we need to get rid of the root sign. The strategy is to reformulate the equation so that the root sign only appears on one side of the equals sign. Then we can square both sides of the equation (in the case of quadratic roots), so that the root sign disappears, and solve the resulting (squared) equation. We have to be careful though, since a solution of the resulting equation might not be a solution of the original equation. We lose information when squaring, since both positive and negative quantities become positive. Therefore there may be values that satisfy the squared equation but not the original equation. These extra roots are called spurious roots. We must always identify any spurious roots by checking whether or not each of the solutions we have obtained satisfies the original equation.

Example 1

Consider a simple (trivial) equation

$$x = 2.$$

If we square both sides of this equation, we get

$$x^2 = 4.$$

This new equation has two solutions $x = 2$ or $x = -2$. The solution $x = 2$ satisfies the original equation, while $x = -2$ only satisfies the squared equation. So we see that the squared equation has more solutions than the original equation. In this case $x = -2$ is a spurious root.

Example 2

Solve the equation $2\sqrt{x-1} = 1 - x$.

If we square both sides of the equation we get

$$4(x-1) = (1-x)^2.$$

Expanding the square on the right-hand side we see that

$$4(x-1) = 1 - 2x + x^2.$$

This is a quadratic equation, which can be written as

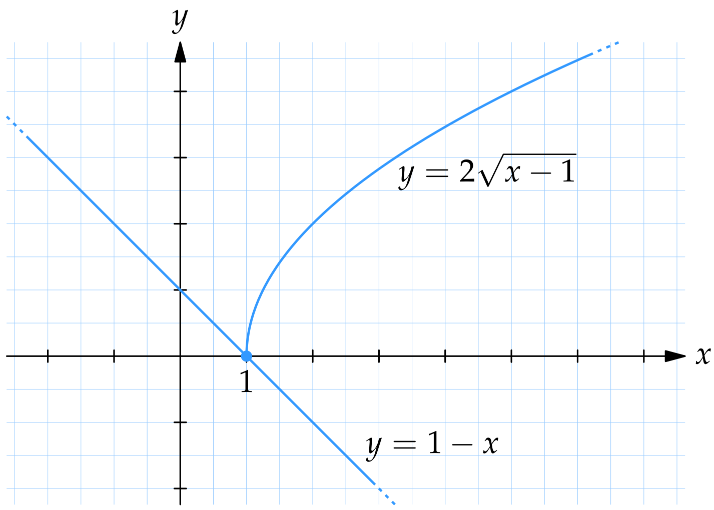
$$x^2 - 6x + 5 = 0.$$

This can be solved by completing the square or by using the general solution formula. Either way the solutions are $x = 3 \pm 2$, i.e. $x = 1$ or $x = 5$.

Since we squared the original equation, there is a risk that spurious roots have been introduced, and therefore we need to check whether $x = 1$ and $x = 5$ are also solutions of the original equation:

- $x = 1$ gives that LHS $= 2\sqrt{1-1} = 0$ and RHS $= 1 - 1 = 0$. So LHS $=$ RHS and the equation is satisfied!
- $x = 5$ gives that LHS $= 2\sqrt{5-1} = 2 \cdot 2 = 4$ and RHS $= 1 - 5 = -4$. So LHS $\neq$ RHS and the equation is not satisfied!

Thus the equation has only one solution $x = 1$.



Study advice

The basic and final tests
After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that...
When squaring an equation remember that the solutions obtained might not all be solutions of the original equation; that is, we might generate spurious roots. This is because we lose information when squaring, since minus signs disappear. Therefore, one must verify that the solutions obtained are indeed solutions of the original equation.

You should always test the solution in the original equation.

Reviews
For those of you who want to deepen your understanding or need more detailed explanations consider the following reference:

- Understanding Algebra - English online book for pre-university studies (`http://www.jamesbrennan.org/algebra/`)

Useful web sites

- Webmath.com can help you to simplify root expressions. (`http://www.webmath.com/simpsqrt.html`)

3.2 Exercises

Exercise 3.2:1

Solve the following equation $\sqrt{x-4} = 6 - x$.

Exercise 3.2:2

Solve the following equation $\sqrt{2x+7} = x + 2$.

Exercise 3.2:3

Solve the following equation $\sqrt{3x-8} + 2 = x$.

Exercise 3.2:4

Solve the following equation $\sqrt{1-x} = 2 - x$.

Exercise 3.2:5

Solve the following equation $\sqrt{3x-2} = 2 - x$.

Exercise 3.2:6

Solve the following equation $\sqrt{x+1} + \sqrt{x+5} = 4$.

3.3 Logarithms

Contents:

- Logarithms
- Fundamental Laws of Logarithms

Learning outcomes:

After this section, you will have learned:

- The concepts of base and exponent.
- The meaning of the notation ln, lg, log and $\log_a$.
- To calculate simple logarithmic expressions using the definition of a logarithm.
- That logarithms are only defined for positive numbers.
- The value of the number e.
- To use the laws of logarithms to simplify logarithmic expressions.
- To know when the laws of logarithms are valid.
- To express a logarithm in terms of a logarithm with a different base.

Logarithms to the base 10

We often use powers with base 10 to represent large and small numbers, for example

$$10^3 = 10 \cdot 10 \cdot 10 = 1000,$$
$$10^{-2} = \frac{1}{10 \cdot 10} = \frac{1}{100} = 0{,}01.$$

Informally, we might say that that

> "the exponent of 1000 is 3", oder dass
> "the exponent of 0.01 is -2".

In fact, though, the term *logarithm* is used instead, and we say:

> "*The logarithm* of 1000 is 3",

which is written as $\lg 1000 = 3$, and

"*The logarithm* of 0.01 is -2",

which is written as $\lg 0.01 = -2$.

More generally, we say:

The logarithm of a number y is denoted by $\lg y$ and is the real number x in the blue box which satisfies the equality

$$10^{\boxed{}} = y.$$

One way to think about this is that the logarithm of y is the answer to the question "10 to the power *what* is equal to y?" Thus, $\lg 100$ is the answer to the question "10 to the power *what* is equal to 1000?"; this answer is clearly 3.

Note that y must be a positive number for the logarithm $\lg y$ to be defined, since there is no power of 10 that evaluates to a negative number or zero.

Example 1

a) $\lg 100000 = 5$ because $10^{\boxed{5}} = 100\,000$.

b) $\lg 0{,}0001 = -4$ because $10^{\boxed{-4}} = 0{,}0001$.

c) $\lg \sqrt{10} = \frac{1}{2}$ because $10^{\boxed{1/2}} = \sqrt{10}$.

d) $\lg 1 = 0$ because $10^{\boxed{0}} = 1$.

e) $\lg 10^{78} = 78$ because $10^{\boxed{78}} = 10^{78}$.

f) $\lg 50 \approx 1{,}699$ because $10^{\boxed{1{,}699}} \approx 50$.

g) $\lg(-10)$ does not exist because $10^{\boxed{a}}$ can never be -10 regardless of how a is chosen.

In the second-to-last example, one can easily understand that $\lg 50$ must lie somewhere between 1 and 2 since $10^1 < 50 < 10^2$. However, in practice to obtain a more precise value of the irrational number $\lg 50 = 1.69897\ldots$ one needs a calculator (or table).

Example 2

a) $10^{\lg 100} = 100$

b) $10^{\lg a} = a$

c) $10^{\lg 50} = 50$

Different bases

We can of course work with powers of numbers other than 10, answering questions like "2 to the power *what* is equal to 8?" The answers to questions of this type are also called logarithms, except *to a different base*. In our example, we would say that the logarithm to base 2 of 8 is 3.

We write $\log_2$ to mean a logarithm with base 2, and so on for other bases.

Example 3

a) $\log_2 8 = 3$ because $2^{3} = 8$.

b) $\log_2 2 = 1$ because $2^{1} = 2$.

c) $\log_2 1024 = 10$ because $2^{10} = 1024$.

d) $\log_2 \frac{1}{4} = -2$ because $2^{-2} = \dfrac{1}{2^2} = \dfrac{1}{4}$.

We deal with bases like 3 or 5 in the same way.

Example 4

a) $\log_3 9 = 2$ because $3^{2} = 9$.

b) $\log_5 125 = 3$ because $5^{3} = 125$.

c) $\log_4 \dfrac{1}{16} = -2$ because $4^{-2} = \dfrac{1}{4^2} = \dfrac{1}{16}$.

d) $\log_b \dfrac{1}{\sqrt{b}} = -\dfrac{1}{2}$ because $b^{-1/2} = \dfrac{1}{b^{1/2}} = \dfrac{1}{\sqrt{b}}$ (if $b > 0$ and $b \neq 1$).

If the base 10 is being used, one rarely writes $\log_{10}$ except for emphasis. Instead, the notation lg is used, or sometimes simply log (beware, however: the notation log can also be used in place of ln, which you are about to meet; the meaning of log on its own often depends on the context). These symbols appear on many calculators.

Note that it makes no sense to define logarithms with base 1, since 1 to the power anything is 1.

The natural logarithms

In practice there are two bases that are commonly used for logarithms, 10 and the number e ($\approx 2.71828\ldots$). (In some mathematical contexts, logarithms to base 2 are also used.) Logarithms using the base e are called *natural logarithms* and we use the notation ln instead of $\log_e$.

Example 5

a) $\ln 10 \approx 2{,}3$ because $e^{2{,}3} \approx 10$.

b) $\ln e = 1$ because $e^{1} = e$.

c) $\ln \dfrac{1}{e^3} = -3$ because $e^{-3} = \dfrac{1}{e^3}$.

d) $\ln 1 = 0$ because $e^{0} = 1$.

e) If $y = e^a$ then $a = \ln y$.

f) $e^{\ln 5} = 5$

g) $e^{\ln x} = x$

Most advanced calculators have buttons for 10-logarithms and natural logarithms.

The reason e is such an important base for logarithms will not really become clear until the second part of this course, when you study differentiation. In the meantime, please bear with us; "natural logarithms" might seem strange, but they really do turn out to be "natural".

Laws of Logarithms

Between the years 1617 and 1624 Henry Biggs published a table of logarithms for all integers up to 20 000, and in 1628 Adriaan Vlacq expanded the table for all integers up to 100 000. The reason such an enormous amount of work was invested in producing these tables is that with the help of logarithms one can multiply numbers together just by adding their logarithms (addition is a much faster calculation than multiplication).

Example 6

Calculate $35 \cdot 54$.

If we know that $35 \approx 10^{1.5441}$ and $54 \approx 10^{1.7324}$ (i.e. $\lg 35 \approx 1.5441$ and $\lg 54 \approx 1.7324$) then we can calculate that

$$35 \cdot 54 \approx 10^{1,5441} \cdot 10^{1,7324} = 10^{1,5441+1,7324} = 10^{3,2765}.$$

Since $10^{3.2765} \approx 1890$ (i.e. $\lg 1890 \approx 3.2765$), we have thus managed to calculate the product

$$35 \cdot 54 = 1890$$

just by adding together the exponents 1.5441 and 1.7324.

In the above example we have used a logarithmic law which states that

$$\log(ab) = \log a + \log b,$$

for all $a, b > 0$. We can see that this is true by using the laws of exponents. Indeed,

$$a \cdot b = 10^{\log a} \cdot 10^{\log b} = 10^{\log a + \log b}$$

and on the other hand,

$$a \cdot b = 10^{\log(ab)},$$

so that we can see $\log(ab) = \log a + \log b$. By exploiting the laws of exponents in this way we can obtain the corresponding *laws of logarithms*:

$$\log(ab) = \log a + \log b,$$
$$\log \frac{a}{b} = \log a - \log b,$$
$$\log a^b = b \cdot \log a.$$

for $a, b > 0$. These laws hold regardless of the base we are working with.

Example 7

a) $\lg 4 + \lg 7 = \lg(4 \cdot 7) = \lg 28$

b) $\lg 6 - \lg 3 = \lg \frac{6}{3} = \lg 2$

c) $2 \cdot \lg 5 = \lg 5^2 = \lg 25$

d) $\lg 200 = \lg(2 \cdot 100) = \lg 2 + \lg 100 = \lg 2 + 2$

Example 8

a) $\lg 9 + \lg 1000 - \lg 3 + \lg 0{,}001 = \lg 9 + 3 - \lg 3 - 3 = \lg 9 - \lg 3 = \lg \dfrac{9}{3} = \lg 3$

b) $$\ln \frac{1}{e} + \ln \sqrt{e} = \ln\Big(\frac{1}{e} \cdot \sqrt{e}\Big) = \ln\Big(\frac{1}{(\sqrt{e}\,)^2} \cdot \sqrt{e}\Big) = \ln \frac{1}{\sqrt{e}}$$
$$= \ln e^{-1/2} = -\tfrac{1}{2} \cdot \ln e = -\tfrac{1}{2} \cdot 1 = -\tfrac{1}{2}$$

c) $$\begin{aligned} \log_2 36 - \tfrac{1}{2}\log_2 81 &= \log_2(6 \cdot 6) - \tfrac{1}{2}\log_2(9 \cdot 9) \\ &= \log_2(2 \cdot 2 \cdot 3 \cdot 3) - \tfrac{1}{2}\log_2(3 \cdot 3 \cdot 3 \cdot 3) \\ &= \log_2(2^2 \cdot 3^2) - \tfrac{1}{2}\log_2(3^4) \\ &= \log_2 2^2 + \log_2 3^2 - \tfrac{1}{2}\log_2 3^4 \\ &= 2\log_2 2 + 2\log_2 3 - \tfrac{1}{2} \cdot 4\log_2 3 \\ &= 2 \cdot 1 + 2\log_2 3 - 2\log_2 3 = 2 \end{aligned}$$

d) $$\begin{aligned} \lg a^3 - 2\lg a + \lg \frac{1}{a} &= 3\lg a - 2\lg a + \lg a^{-1} \\ &= (3-2)\lg a + (-1)\lg a = \lg a - \lg a = 0 \end{aligned}$$

Changing the base

It is sometimes a good idea to express a logarithm as a logarithm with respect to another base.

Example 9

a) Express $\lg 5$ as a natural logarithm.

By definition, $\lg 5$ is a number that satisfies the equality

$$10^{\lg 5} = 5$$

Taking the natural logarithm (ln) of both sides yields

$$\ln 10^{\lg 5} = \ln 5.$$

With the help of the logarithm law $\ln a^b = b \ln a$, the left-hand side can be written as $\lg 5 \times \ln 10$ and the equality becomes

$$\lg 5 \cdot \ln 10 = \ln 5.$$

Now divide both sides by $\ln 10$ to get the answer

$$\lg 5 = \frac{\ln 5}{\ln 10} \qquad (\approx 0{,}699\,, \quad \text{also ist } 10^{0{,}699} \approx 5).$$

b) Express the 2-logarithm of 100 as a 10-logarithm lg.

Using the definition of a logarithm one has that $\log_2 100$ formally satisfies

$$2^{\log_2 100} = 100$$

Taking the 10-logarithm of both sides, we get

$$\lg 2^{\log_2 100} = \lg 100.$$

Since $\lg a^b = b \lg a$ we get $\lg 2^{\log_2 100} = \log_2 100 \times \lg 2$, and moreover the right-hand side can be simplified to $\lg 100 = 2$. Thus we see that

$$\log_2 100 \cdot \lg 2 = 2.$$

Finally, dividing by $\lg 2$ gives

$$\log_2 100 = \frac{2}{\lg 2} \qquad (\,\approx 6.64\,, \quad \text{therefore } 2^{6.64} \approx 100).$$

The general formula for changing from one base a to another base b is

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

and can be derived in the same way. We can also change the base of a power using logarithms. For instance, if we want to write 2^5 using the base 10 first write 2 as a power with the base 10:

$$2 = 10^{\lg 2},$$

Then, using one of the laws of exponents,

$$2^5 = (10^{\lg 2})^5 = 10^{5 \cdot \lg 2} \quad (\,\approx 10^{1{,}505}\,).$$

Example 10

a) Write 10^x using the base e.

First, we write 10 as a power of e,

$$10 = e^{\ln 10}.$$

Using the the laws of exponents we can then see that

$$10^x = (e^{\ln 10})^x = e^{x \cdot \ln 10} \approx e^{2{,}3x}.$$

b) Write e^a using the base 10.

The number e can be written as $e = 10^{\lg e}$ and therefore

$$e^a = (10^{\lg e})^a = 10^{a \cdot \lg e} \approx 10^{0{,}434a}.$$

Study advice

The basic and final tests
After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that ...
You may need to spend some time studying logarithms.

Logarithms are not usually dealt with in detail in secondary school, but become more important at university; this can cause problems.

Reviews
For those of you who want to deepen your understanding or need more detailed explanations consider the following references:

- Learn more about logarithms on Wikipedia
  (`http://en.wikipedia.org/wiki/Logarithm`)
- Learn more about the number e in The MacTutor History of Mathematics archive
  (`http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/e.html`)

Useful web sites

- Experiment with logarithms and powers
(http://www.ltcconline.net/greenl/java/IntermedCollegeAlgebra/LogGraph/logGraph.html)
- Play logarithm Memory
(http://www.ltcconline.net/greenl/java/IntermedCollegeAlgebra/LogConcentration/LogConcentration.htm)
- Help the frog to jump onto his water-lily leaf in the "log" game
(http://www.ltcconline.net/greenl/java/IntermedCollegeAlgebra/logger.htm)

3.3 Exercises

Exercise 3.3:1

Solve the following equations for x.

a) $10^x = 1\,000$

b) $10^x = 0{,}1$

c) $\dfrac{1}{10^x} = 100$

d) $\dfrac{1}{10^x} = 0{,}000\,1$

Exercise 3.3:2

Calculate

a) $\lg 0{,}1$

b) $\lg 10\,000$

c) $\lg 0{,}001$

d) $\lg 1$

e) $10^{\lg 2}$

f) $\lg 10^3$

g) $10^{-\lg 0{,}1}$

h) $\lg \dfrac{1}{10^2}$

Exercise 3.3:3

Calculate

a) $\log_2 8$

b) $\log_9 \dfrac{1}{3}$

c) $\log_2 0{,}125$

d) $\log_3 \left(9 \cdot 3^{1/3}\right)$

e) $2^{\log_2 4}$

f) $\log_2 4 + \log_2 \dfrac{1}{16}$

g) $\log_3 12 - \log_3 4$

h) $\log_a \left(a^2 \sqrt{a}\,\right)$

Exercise 3.3:4

Simplify

a) $\lg 50 - \lg 5$

b) $\lg 23 + \lg \dfrac{1}{23}$

c) $\lg 27^{1/3} + \dfrac{\lg 3}{2} + \lg \dfrac{1}{9}$

Exercise 3.3:5

Simplify

a) $\ln e^3 + \ln e^2$

b) $\ln 8 - \ln 4 - \ln 2$

c) $(\ln 1) \cdot e^2$

d) $\ln e - 1$

e) $\ln \dfrac{1}{e^2}$

f) $\left(e^{\ln e}\right)^2$

Exercise 3.3:6

Use the calculator on the right to calculate the following to three decimal places. The button `LN` signifies the natural logarithm with base e.

a) $\log_3 4$

b) $\lg 46$

c) $\log_3 \log_2 \left(3^{118}\right)$

3.4 Logarithmic equations

Contents:

- Logarithmic equations
- Exponential equations
- Spurious roots

Learning outcomes:

After this section, you will have learned to:

- Solve equations, containing logarithmic or exponential expressions, that can be reduced to a linear or quadratic form.
- Deal with spurious roots, and know when they arise.
- Determine which of two logarithmic expressions is the largest by means of a comparison of bases argument.

Basic Equations

Equations involving logarithms can vary a lot. Here are two simple examples which we can solve straight away using the definition of the logarithm:

$$\begin{aligned} 10^x = y \quad &\Leftrightarrow \quad x = \lg y, \\ e^x = y \quad &\Leftrightarrow \quad x = \ln y. \end{aligned}$$

In this section, we consider only logarithms to base 10 or natural logarithms (logarithms to base e), though the methods can just as easily be applied in the case of logarithms with an arbitrary base.

Example 1

We solve the following equations for x:

a) $10^x = 537$ has a solution $x = \lg 537$.

b) $10^{5x} = 537$ gives $5x = \lg 537$, i.e. $x = \frac{1}{5}\lg 537$.

c) $\dfrac{3}{e^x} = 5$ Multiplying both sides by e^x and division by 5 gives $\frac{3}{5} = e^x$, which means that $x = \ln \frac{3}{5}$.

d) $\lg x = 3$. The definition of logarithm then directly gives$x = 10^3 = 1000$.

e) $\lg(2x - 4) = 2$. From the definition of logarithm we have $2x - 4 = 10^2 = 100$ and it follows that $x = 52$.

Example 2

a) Solve the equation $(\sqrt{10}\,)^x = 25$.

Since $\sqrt{10} = 10^{1/2}$, the left-hand side is equal to $(\sqrt{10}\,)^x = (10^{1/2})^x = 10^{x/2}$ and the equation becomes

$$10^{x/2} = 25.$$

This equation has solution $\frac{x}{2} = \lg 25$, ie. $x = 2\lg 25$.

b) Solve the equation $\dfrac{3\ln 2x}{2} + 1 = \dfrac{1}{2}$.

Multiply both sides by 2 and then subtract 2 from both sides to get

$$3\ln 2x = -1.$$

Dividing both sides by 3 gives

$$\ln 2x = -\frac{1}{3}.$$

Now, the definition of logarithm directly gives $2x = e^{-1/3}$, so that

$$x = \tfrac{1}{2}\,e^{-1/3} = \frac{1}{2\,e^{1/3}}.$$

In many practical applications of exponential growth or decay there appear equations of the type

$$a^x = b,$$

where a and b are positive numbers. These equations are best solved by taking the logarithm of both sides so that

$$\lg a^x = \lg b.$$

Then by the laws of logarithms,

$$x \cdot \lg a = \lg b,$$

which gives the solution $x = \dfrac{\lg b}{\lg a}$.

Example 3

a) Solve the equation $3^x = 20$.

Take logarithms of both sides to get

$$\lg 3^x = \lg 20.$$

The left-hand side can be written as $\lg 3^x = x \lg 3$ giving

$$x = \frac{\lg 20}{\lg 3} \quad (\approx 2.727).$$

b) Solve the equation $5000 \times 1.05^x = 10\,000$.

Divide both sides by 5 000 to get

$$1.05^x = \frac{10\,000}{5\,000} = 2.$$

This equation can be solved by taking the lg logarithm of both sides of and rewriting the left-hand side as $\lg 1.05^x = x \cdot \lg 1.05$. Then

$$x = \frac{\lg 2}{\lg 1.05} \quad (\approx 14.2).$$

Example 4

a) Solve the equation $2^x \cdot 3^x = 5$.

The left-hand side can be rewritten using the laws of indices to give $2^x \cdot 3^x = (2 \cdot 3)^x$ and the equation becomes

$$6^x = 5.$$

This equation is solved in the usual way by taking logarithms, giving

$$x = \frac{\lg 5}{\lg 6} \quad (\approx 0.898).$$

b) Solve the equation $5^{2x+1} = 3^{5x}$.

Take logarithms of both sides and use the laws of logarithms to get

$$\begin{aligned}(2x+1)\lg 5 &= 5x \cdot \lg 3,\\ 2x \cdot \lg 5 + \lg 5 &= 5x \cdot \lg 3.\end{aligned}$$

Collecting x on one side gives

$$\begin{aligned}\lg 5 &= 5x \cdot \lg 3 - 2x \cdot \lg 5,\\ \lg 5 &= x\,(5\lg 3 - 2\lg 5).\end{aligned}$$

The solution is then

$$x = \frac{\lg 5}{5\lg 3 - 2\lg 5}.$$

Some more complicated equations

Equations containing exponential or logarithmic expressions can sometimes be treated as linear or quadratic equations by considering "$\ln x$" or "e^x" as the unknown variable.

Example 5

Solve the equation $\dfrac{6e^x}{3e^x+1} = \dfrac{5}{e^{-x}+2}$.

Multiply both sides by $3e^x + 1$ and $e^{-x} + 2$ to eliminate the denominators, so that

$$6e^x(e^{-x}+2) = 5(3e^x+1).$$

In this last step we have multiplied the equation by factors $3e^x + 1$ and $e^{-x} + 2$. Neither of these factors can possibly be zero, so we are safe; however, if either can be zero, there is a danger that spurious roots may thus be introduced.

Simplify both sides of the equation to get

$$6 + 12e^x = 15e^x + 5.$$

Here we have used $e^{-x} \times e^x = e^{-x+x} = e^0 = 1$. If we treat e^x as the unknown variable, the equation is essentially a linear equation which has a solution

$$e^x = \frac{1}{3}.$$

Taking logarithms then gives the answer:

$$x = \ln\frac{1}{3} = \ln 3^{-1} = -1 \cdot \ln 3 = -\ln 3.$$

Example 6

Solve the equation $\dfrac{1}{\ln x} + \ln\dfrac{1}{x} = 1.$

The term $\ln\frac{1}{x}$ can be written as $\ln\frac{1}{x} = \ln x^{-1} = -1 \times \ln x = -\ln x$, and then the equation becomes

$$\frac{1}{\ln x} - \ln x = 1,$$

We multiply both sides by $\ln x$ (which is non-zero when $x \neq 1$, and $x = 1$ is clearly not a solution) and this gives us a quadratic equation in $\ln x$:

$$\begin{aligned} 1 - (\ln x)^2 &= \ln x, \\ (\ln x)^2 + \ln x - 1 &= 0. \end{aligned}$$

Completing the square on the left-hand side we see that

$$\begin{aligned} (\ln x)^2 + \ln x - 1 &= \bigl(\ln x + \tfrac{1}{2}\bigr)^2 - \bigl(\tfrac{1}{2}\bigr)^2 - 1 \\ &= \bigl(\ln x + \tfrac{1}{2}\bigr)^2 - \tfrac{5}{4}. \end{aligned}$$

Then by taking roots,

$$\ln x = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}.$$

This means that the equation has two solutions

$$x = e^{(-1+\sqrt{5})/2} \quad \text{und} \quad x = e^{-(1+\sqrt{5})/2}.$$

Spurious roots

When you solve equations you should also bear in mind that the arguments of logarithms have to be positive and that terms of the type $e^{(\cdots)}$ can only have positive values. In other words we must be careful to make sure that our answer makes sense.

Example 7

Solve the equation $\ln(4x^2 - 2x) = \ln(1 - 2x)$.

For the equation to be satisfied the arguments $4x^2 - 2x$ and $1 - 2x$ must be be positive and equal i.e.

$$4x^2 - 2x = 1 - 2x. \tag{$*$}$$

We solve the equation $(*)$ by moving all of the terms to one side

$$4x^2 - 1 = 0$$

and taking the root. This gives

$$x = -\tfrac{1}{2} \quad \text{and} \quad x = \tfrac{1}{2}.$$

We now check if both sides of $(*)$ are positive

- If $x = -\frac{1}{2}$, then both are sides are equal to $4x^2 - 2x = 1 - 2x = 1 - 2 \cdot \left(-\frac{1}{2}\right) = 1 + 1 = 2 > 0$.
- If $x = \frac{1}{2}$, then both are sides are equal to $4x^2 - 2x = 1 - 2x = 1 - 2 \cdot \frac{1}{2} = 1 - 1 = 0 \not> 0$.

So this logarithmic equation has only one solution $x = -\frac{1}{2}$.

Example 8

Solve the equation $e^{2x} - e^x = \frac{1}{2}$.

The first term can be written as $e^{2x} = (e^x)^2$. The whole equation is a quadratic with e^x as the unknown i.e.

$$(e^x)^2 - e^x = \tfrac{1}{2}.$$

The equation can be a little easier to manage if we write t instead of e^x, so that we try and solve

$$t^2 - t = \tfrac{1}{2}.$$

Completing the square on the left-hand side gives

$$\begin{aligned}\left(t - \tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 &= \tfrac{1}{2}, \\ \left(t - \tfrac{1}{2}\right)^2 &= \tfrac{3}{4},\end{aligned}$$

so that

$$t = \frac{1}{2} - \frac{\sqrt{3}}{2} \quad \text{und} \quad t = \frac{1}{2} + \frac{\sqrt{3}}{2}.$$

Since $\sqrt{3} > 1$, $\frac{1}{2} - \frac{1}{2}\sqrt{3} < 0$. Therefore it is only $t = \frac{1}{2} + \frac{1}{2}\sqrt{3}$ that provides a solution of the original equation because e^x is always positive. Taking logarithms finally gives

$$x = \ln\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$

as the only solution of the equation.

Study advice

The basic and final tests
After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that:
You may need to spend some time studying logarithms.

Logarithms are not usually dealt with in detail in secondary school, but become important at university. This can cause problems.

3.4 Exercises

Exercise 3.4:1

Solve the equation

a) $e^x = 13$ b) $13e^x = 2 \cdot 3^{-x}$ c) $3e^x = 7 \cdot 2^x$

Exercise 3.4:2

Solve the equation

a) $2^{x^2-2} = 1$ b) $e^{2x} + e^x = 4$ c) $3e^{x^2} = 2^x$

Exercise 3.4:3

Solve the equation

a) $2^{-x^2} = 2e^{2x}$ b) $\ln(x^2 + 3x) = \ln(3x^2 - 2x)$

c) $\ln x + \ln(x+4) = \ln(2x+3)$

4.1 Angles and circles

Contents:

- Various angle measures (degrees and radians)
- Pythagoras' theorem
- Formula for distance in the plane
- Equation of a circle

Learning outcomes:

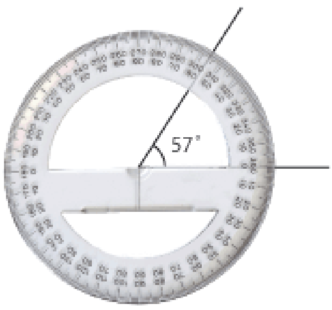
After this section, you will have learned:

- To convert between degree and radians.
- To calculate the area and circumference of sectors of a circle.
- The features of right-angled triangles.
- To formulate and use Pythagoras' theorem.
- To calculate the distance between two points in the plane.
- To sketch circles by completing the square.
- The concepts of the unit circle, tangent, radius, diameter, circumference, chord and arc.
- To solve geometric problems that contain circles.

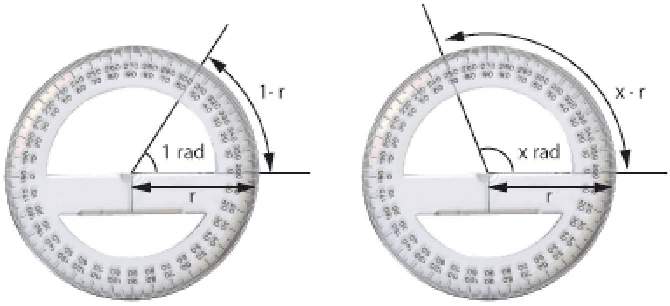
Angle measures

There are several different units for measuring angles, which are used in different contexts. The two most common within mathematics are degrees and radians.

- **Degrees.** If a complete revolution is divided into 360 parts, then each part is called 1 degree. Degrees are designated by $^\circ$.



- **Radians.** Another way to measure an angle is to consider an arc described by the angle, and simply divide the length of this arc by its radius. This unit is called the radian. A full circle is 2π radians, since the circumference of a circle is $2\pi r$, where r is its radius.



A complete circle is 360° or 2π radians which means

$$1^\circ = \frac{1}{360} \cdot 2\pi \text{ radians } = \frac{\pi}{180} \text{ radians,}$$
$$1 \text{ radian } = \frac{1}{2\pi} \cdot 360^\circ = \frac{180^\circ}{\pi}.$$

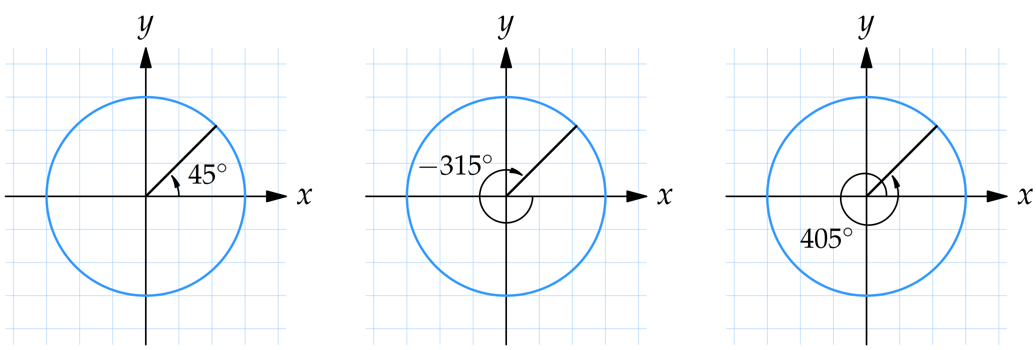
These conversion relations can be used to convert between degrees and radians.

Example 1

a) $30^\circ = 30 \cdot 1^\circ = 30 \cdot \dfrac{\pi}{180} \text{ rad} = \dfrac{\pi}{6} \text{ rad}$

b) $\dfrac{\pi}{8} \text{ radians} = \dfrac{\pi}{8} \cdot (1 \text{ rad}) = \dfrac{\pi}{8} \cdot \dfrac{180^\circ}{\pi} = 22{,}5^\circ$

In some contexts, it may be useful to talk about negative angles and angles greater than 360°. This means that the same point on the circle can be designated by different angles that differ from each other by a whole number of complete revolutions.



Example 2

a) The angles $-55°$ and $665°$ indicate the same point on the circle because

$$-55° + 2 \cdot 360° = 665°.$$

b) The angles $\dfrac{3\pi}{7}$ and $-\dfrac{11\pi}{7}$ indicate the same point on the circle because

$$\frac{3\pi}{7} - 2\pi = -\frac{11\pi}{7}.$$

c) The angles $36°$ and $216°$ do not specify the same point on the circle, but opposite points since

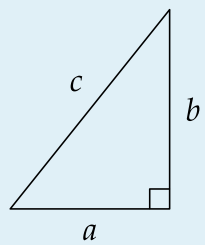
$$36° + 180° = 216°.$$

Formula for distance in the plane

Pythagoras' theorem is one of the most famous statements in the whole of mathematics; it says that in a right-angled triangle with legs (that is, short sides) a and b, and hypotenuse c then $a^2 + b^2 = c^2$.

Pythagoras' theorem:

$$c^2 = a^2 + b^2.$$



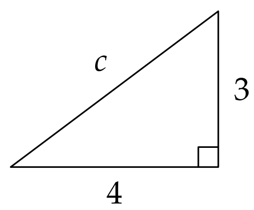
Example 3

Consider the triangle on the right. Then

$$c^2 = 3^2 + 4^2 = 9 + 16 = 25$$

and therefore the hypotenuse c is equal to

$$c = \sqrt{25} = 5.$$

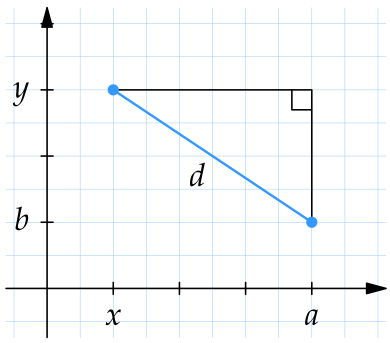


Pythagoras' theorem can be used to calculate the distance between two points in a coordinate system.

Formula for distance:
The distance d between two points with coordinates (x, y) and (a, b) is

$$d = \sqrt{(x-a)^2 + (y-b)^2}.$$

The line joining the points is the hypotenuse of a triangle whose legs are parallel to the coordinate axes.



The legs of the triangle have lengths equal to the difference in the x- and y-directions of the points, that is $|x-a|$ and $|y-b|$. Pythagoras' theorem then gives the formula for the distance.

Example 4

a) The distance between $(1, 2)$ and $(3, 1)$ is

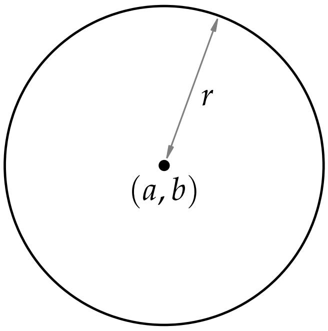
$$d = \sqrt{(1-3)^2 + (2-1)^2} = \sqrt{(-2)^2 + 1^2} = \sqrt{4+1} = \sqrt{5}.$$

b) The distance between $(-1, 0)$ and $(-2, -5)$ is

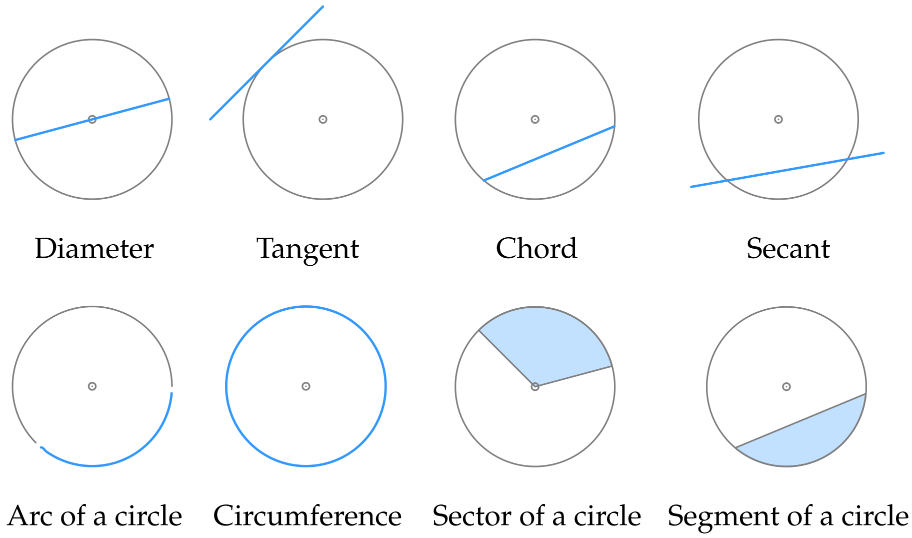
$$d = \sqrt{(-1-(-2))^2 + (0-(-5))^2} = \sqrt{1^2 + 5^2} = \sqrt{1+25} = \sqrt{26}.$$

Circles

A circle consists of all the points that are at a given fixed distance r from a point (a, b).



The distance r is called the circle's radius and the point (a, b) is its centre. The figure below shows the other important concepts.



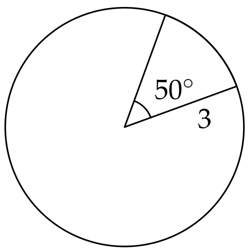
Example 5

A sector of a circle is shown in the figure on the right.

a) Determine its arc length.

The central angle is 50°. In radians this is

$$50^\circ = 50 \cdot 1^\circ = 50 \cdot \frac{\pi}{180} \text{ rad} = \frac{5\pi}{18} \text{ rad}.$$



The way radians have been defined means that the arc length is simply the radius multiplied by the angle measured in radians, so that the arc length is

$$3 \cdot \frac{5\pi}{18} \text{ units} = \frac{5\pi}{6} \text{ units}.$$

b) Determine the area of the circle segment.

The segment's share of the entire circle is

$$\frac{50^\circ}{360^\circ} = \frac{5}{36}$$

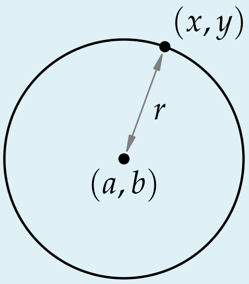
and this means that its area is $\frac{5}{36}$ parts of the circle area, which is $\pi r^2 = \pi 3^2 = 9\pi$. Hence the area is

$$\frac{5}{36} \cdot 9\pi \text{ units} = \frac{5\pi}{4} \text{ units.}$$

A point (x, y) lies on the circle that has its centre at (a, b) and radius r, if its distance from the centre is equal to r. This condition can be formulated with the distance formula.

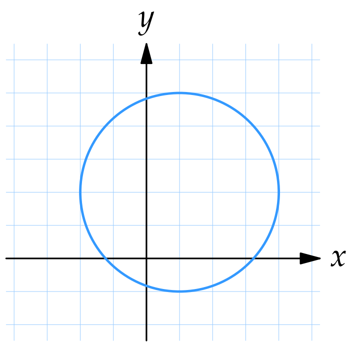
Circle equation:

$$(x-a)^2 + (y-b)^2 = r^2.$$



Example 6

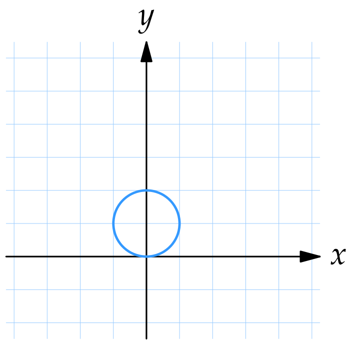
a) $(x-1)^2 + (y-2)^2 = 9$ is the equation for a circle with centre $(1, 2)$ and radius $\sqrt{9} = 3$.



b) $x^2 + (y-1)^2 = 1$: The equation can be written as

$$(x-0)^2 + (y-1)^2 = 1,$$

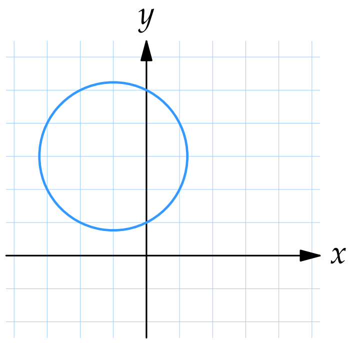
and is the equation of a circle with centre $(0,1)$ and radius $\sqrt{1} = 1$.



c) $(x+1)^2 + (y-3)^2 = 5$: The equation can be written as

$$(x-(-1))^2 + (y-3)^2 = 5,$$

and is the equation of a circle with centre $(-1,3)$ and radius $\sqrt{5} \approx 2.236$.



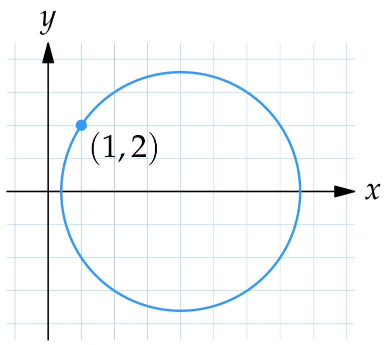
Example 7

a) Does the point $(1,2)$ lie on the circle $(x-4)^2 + y^2 = 13$?

Inserting the coordinates of the point $x = 1$ and $y = 2$ in the circle equation, we have that

$$\text{LHS} = (1-4)^2 + 2^2 = (-3)^2 + 2^2 = 9 + 4 = 13 = \text{RHS}.$$

Since the point satisfies the circle equation it lies on the circle.



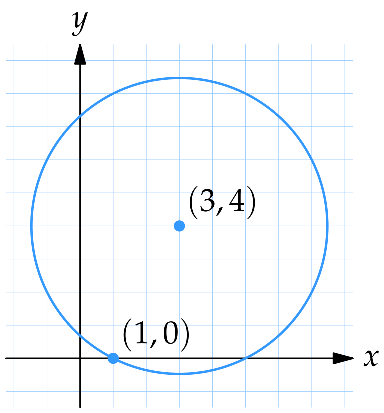
b) Determine the equation for the circle that has its centre at $(3,4)$ and goes through the point $(1,0)$.

Since the point $(1,0)$ lies on the circle, the radius of the circle must be equal to the distance of the point from $(1,0)$ to the centre $(3,4)$. The distance formula allows us to calculate that this distance is

$$c = \sqrt{(3-1)^2 + (4-0)^2} = \sqrt{4+16} = \sqrt{20}.$$

The circle equation is therefore

$$(x-3)^2 + (y-4)^2 = 20\ .$$



Example 8

Determine the centre and radius of the circle with equation $x^2 + y^2\text{ˇ}2x + 4y + 1 = 0$.

Let us try to write the equation in the form

$$(x-a)^2 + (y-b)^2 = r^2$$

because then we can directly read from this that the centre is (a,b) and the radius is r.

Start by completing the square for the terms containing x on the left-hand side

$$\underline{x^2 - 2x} + y^2 + 4y + 1 = \underline{(x-1)^2 - 1^2} + y^2 + 4y + 1.$$

The underlined terms shows the terms involved.

Now complete the square for the terms containing y

$$(x-1)^2 - 1^2 + \underline{y^2 + 4y} + 1 = (x-1)^2 - 1^2 + \underline{(y+2)^2 - 2^2} + 1.$$

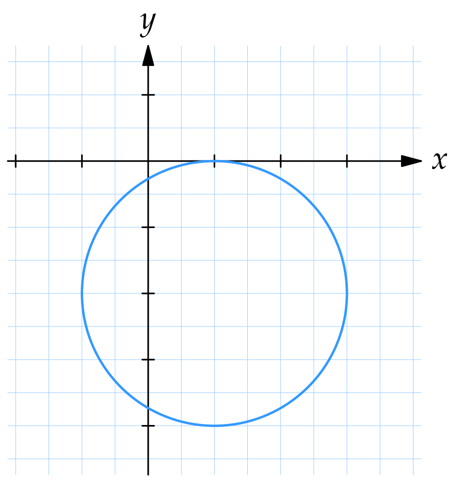
The left-hand side is therefore equal to

$$(x-1)^2 + (y+2)^2 - 4.$$

Rearranging we see that the original equation is equivalent to

$$(x-1)^2 + (y+2)^2 = 4.$$

From this we can easily see that the centre is $(1, -2)$ and the radius is $\sqrt{4} = 2$.



Study advice

The basic and final tests
After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Reviews
For those of you who want to deepen your understanding or need more detailed explanations consider the following references:

- Learn more about Pythagoras theorem from Wikipedia (`http://en.wikipedia.org/wiki/Pythagorean_theorem`)
- Read more about the circle on the Mathworld website (`http://mathworld.wolfram.com/Circle.html`)

Useful web sites

- Interactive experiments: the sine and cosine on the unit circle (`http://www.math.kth.se/online/images/sinus_och_cosinus_i_enhetscirkeln.swf`)

4.1 Exercises

Exercise 4.1:1

Write in degrees and radians

a) $\frac{1}{4}$ revolution

b) $\frac{3}{8}$ revolution

c) $-\frac{2}{3}$ revolution

d) $\frac{97}{12}$ revolution

Exercise 4.1:2

Transform to radians

a) 45°

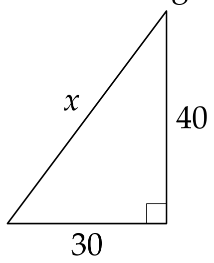
b) 135°

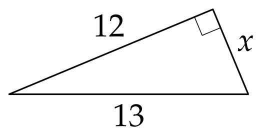
c) -63°

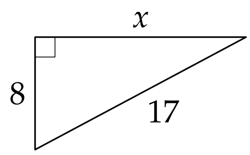
d) 270°

Exercise 4.1:3

Determine the length of the side marked x.

a) 

b) 

c) 

Exercise 4.1:4

a) Determine the distance between the points $(1,1)$ and $(5,4)$.

b) Determine the distance between the points $(-2,5)$ and $(3,-1)$.

c) Find the point on the x-axis which lies as far from the point $(3,3)$ as from $(5,1)$.

Exercise 4.1:5

a) Determine the equation of a circle having its centre at $(1,2)$ and radius 2.

b) Determine the equation of a circle having its centre at $(2,-1)$ and which contains the point $(-1,1)$.

Exercise 4.1:6

Sketch the following circles

a) $x^2+y^2=9$

b) $(x-1)^2+(y-2)^2=3$

c) $(3x-1)^2+(3y+7)^2=10$

Exercise 4.1:7

Sketch the following circles

a) $x^2 + 2x + y^2 - 2y = 1$

b) $x^2 + y^2 + 4y = 0$

c) $x^2 - 2x + y^2 + 6y = -3$

d) $x^2 - 2x + y^2 + 2y = -2$

Exercise 4.1:8

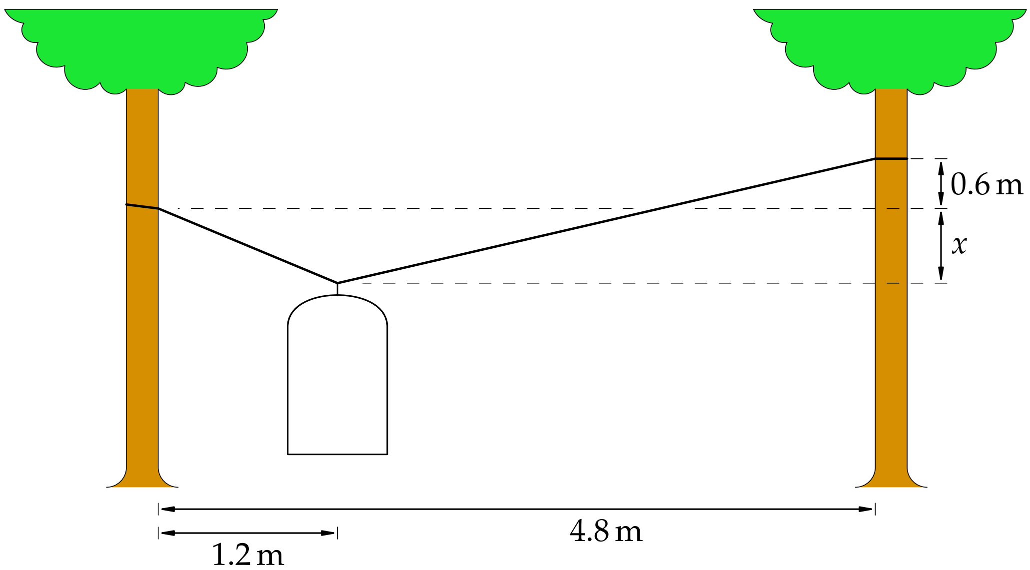
How many revolutions does a wheel of radius 50 cm make when it rolls 10 m?

Exercise 4.1:9

On a clock, the second hand is 8 cm long. How large an area does it sweep through in 10 seconds?

Exercise 4.1:10

A washing line of length 5.4 m hangs between two vertical trees that are at a distance of 4.8 m from each other. One end of the line is fixed 0.6 m higher than the other, and a jacket hangs from a hanger 1.2 m from the tree where the line has its lower point of attachment. Determine how far below the lower attachement point the hanger is hanging. (That is, the distance x in the figure).



4.2 Trigonometric functions

Contents:

- The trigonometric functions cosine, sine and tangent.

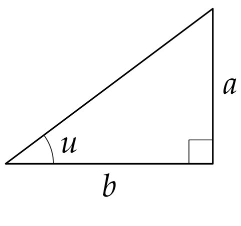
Learning outcomes:

After this section, you will have learned:

- The definition of acute, obtuse and right angles.
- The definition of cosine, sine and tangent.
- The values of cosine, sine and tangent for the standard angles 0, $\pi/6$, $\pi/4$, $\pi/3$ and $\pi/2$ by heart.
- To determine the values of cosine, sine and tangent of arguments that can be reduced to a standard angle.
- To sketch graphs of cosine, sine and tangent.
- To solve trigonometric problems involving right-angled triangles.

Trigonometry of right-angled triangles

In the right-angled triangle below, the ratio between the length a of the side opposite the angle and the length b of the adjacent side is called the tangent of the angle u, and is written as $\tan u$.

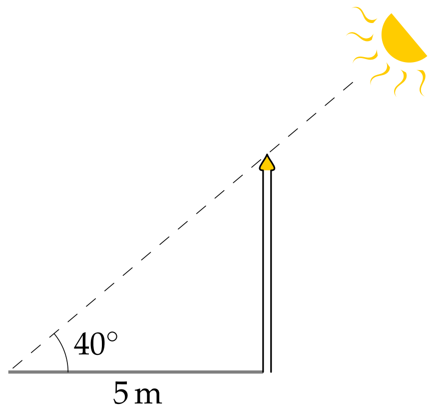


$$\tan u = \frac{a}{b}$$

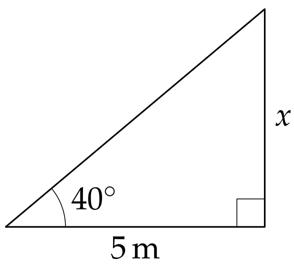
The value of the ratio $\frac{a}{b}$ is not dependent on the size of the triangle, but only on the angle u. For different values of the angle, you can get the value of the tangent either from a trigonometric table or by using a calculator (the relevant button is usually named tan).

Example 1

How high is the flagpole?



The flagpole and its shadow form a right-angled triangle where the vertical side is unknown (marked with x below).



From the definition of tangent, we have that

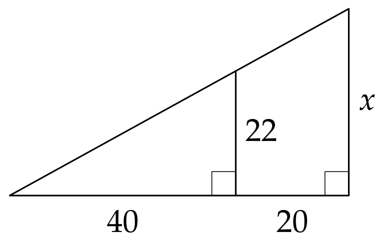
$$\tan 40^\circ = \frac{x}{5\text{ m}}$$

and since $\tan 40^\circ \approx 0.84$ we get

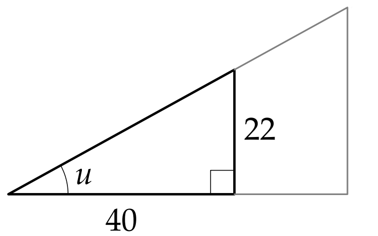
$$x = 5\,\text{m} \cdot \tan 40^\circ \approx 5\,\text{m} \cdot 0{,}84 = 4{,}2\,\text{m}.$$

Example 2

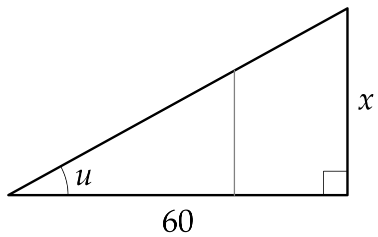
Determine the length of the side designated with the x in the figure.



If we call the angle at the far left u there are two ways to construct an expression for $\tan u$.



$$\tan u = \frac{22}{40}$$



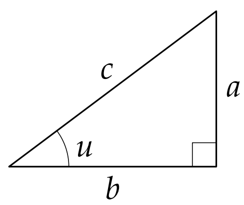
$$\tan u = \frac{x}{60}$$

Equality of the two expressions for $\tan u$ gives

$$\frac{22}{40} = \frac{x}{60}.$$

which leads to $x = 60 \times \dfrac{22}{40} = 33$.

There are two other ratios in right-angled triangles that have special names. The first is $\cos u = b/c$ ("cosine of u") and the second is $\sin u = a/c$ ("sine of u").



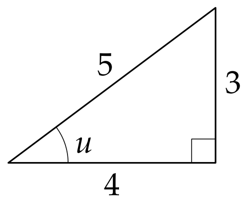
$$\cos u = \frac{b}{c}$$

$$\sin u = \frac{a}{c}$$

Like the tangent the ratios that define the cosine and sine do not depend on the size of the triangle, but only on the angle u.

Example 3

a)

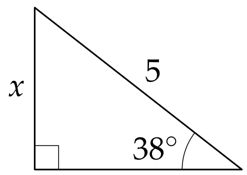


In the triangle on the left

$$\cos u = \frac{4}{5},$$

$$\sin u = \frac{3}{5}.$$

b)



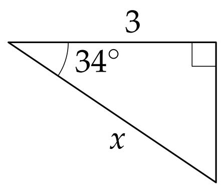
From the definition of sine we have

$$\sin 38^\circ = \frac{x}{5}$$

and if we know that $\sin 38^\circ \approx 0.616$ then

$$x = 5 \cdot \sin 38^\circ \approx 5 \cdot 0.616 \approx 3.1.$$

c)



Cosine is the ratio between the adjacent side and the hypotenuse, so

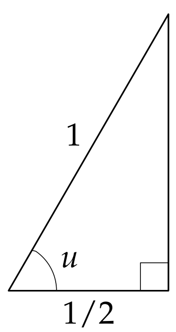
$$\cos 34^\circ = \frac{3}{x}.$$

Thus

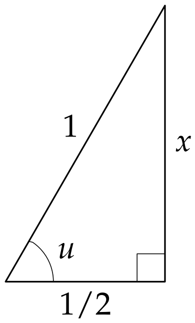
$$x = \frac{3}{\cos 34^\circ}.$$

Example 4

Determine $\sin u$ in the triangle



With the help of Pythagoras' theorem the side on the right can be determined:



$$1^2 = \left(\tfrac{1}{2}\right)^2 + x^2 \quad \Leftrightarrow \quad x = \frac{\sqrt{3}}{2}.$$

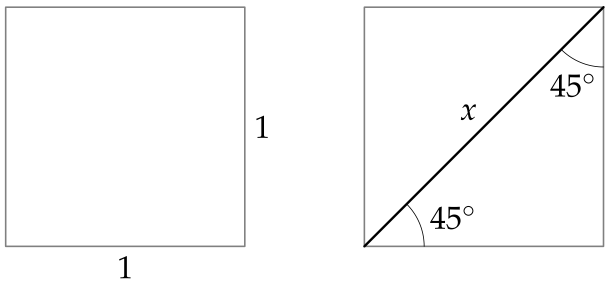
and thus $\sin u = \dfrac{\sqrt{3}/2}{1} = \dfrac{\sqrt{3}}{2}$.

Some standard angles

For some angles, namely 30°, 45° and 60°, it is relatively easy to calculate the exact values of the trigonometric functions.

Example 5

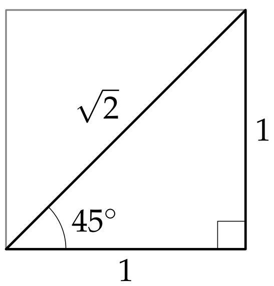
We start with a square having sides of length 1. A diagonal of the square divides the right angles in opposite corners into two equal parts of 45°.



Using Pythagoras' theorem, we can determine the length x of the diagonal:

$$x^2 = 1^2 + 1^2 \quad \Leftrightarrow \quad x = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Each triangle has the diagonal as the hypotenuse. Thus we can obtain the value of the trigonometric functions for the angle 45°:



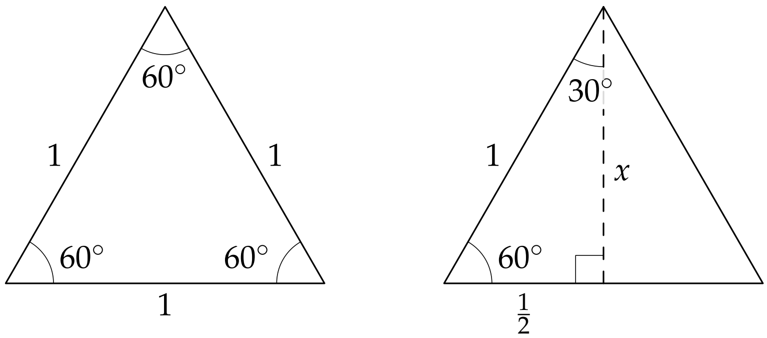
$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

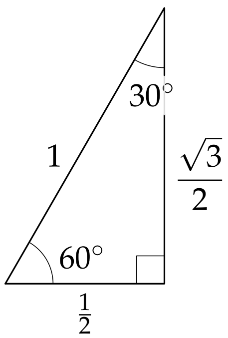
$$\tan 45^\circ = \frac{1}{1} = 1$$

Example 6

Imagine an equilateral triangle where all sides have length 1. The angles of the triangle are all 60°. The triangle can be divided into two halves by a line that divides the angle at the top in equal parts.



The Pythagorean theorem allows us to calculate that the length of the vertical side of half-triangle is $x = \sqrt{3}/2$. Using the definitions we then get that



$$\cos 30^\circ = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}; \qquad \cos 60^\circ = \frac{1/2}{1} = \frac{1}{2}$$

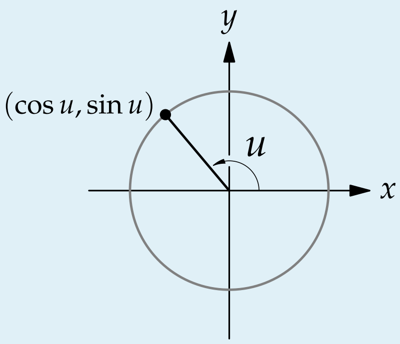
$$\sin 30^\circ = \frac{1/2}{1} = \frac{1}{2}; \qquad \sin 60^\circ = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}; \qquad \tan 60^\circ = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Trigonometric functions for general angles

For angles less than 0° or greater than 90° the trigonometric functions are defined using the unit circle (that is the circle that has its centre at the origin and has radius 1).

The trigonometric functions $\cos u$ and $\sin u$ are the x- and y-coordinates of the point on the unit circle reached by turning through the angle u, as shown in the diagram on the right.



The tangent function is then defined as

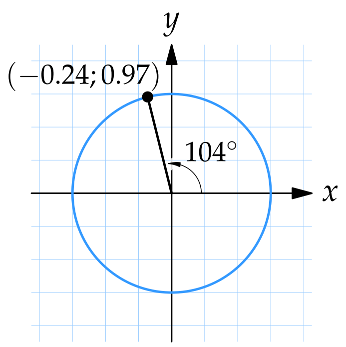
$$\tan u = \frac{\sin u}{\cos u}$$

and the value of the tangent can be interpreted as the gradient of the radius.

Example 7

From the figures below, we obtain the values of cosine and sine.

a)

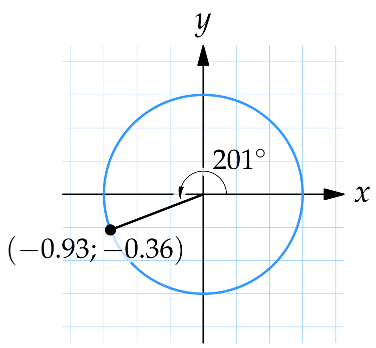


$$\cos 104° \approx -0.24$$

$$\sin 104° \approx 0.97$$

$$\tan 104° \approx \frac{0.97}{-0.24} \approx -4.0$$

b)



$$\cos 201° \approx -0.93$$

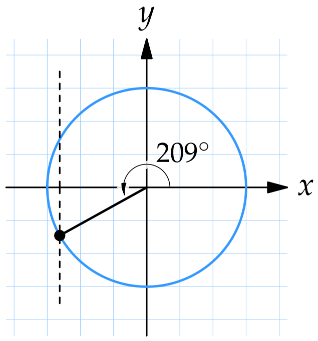
$$\sin 201° \approx -0.36$$

$$\tan 201° \approx \frac{-0.36}{-0.93} \approx 0.4$$

Example 8

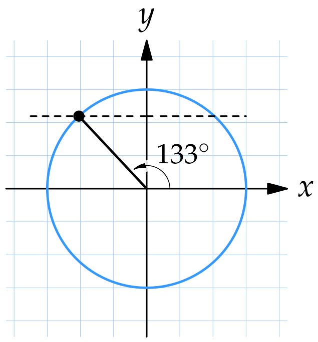
Which sign do the following have?

a) $\cos 209°$



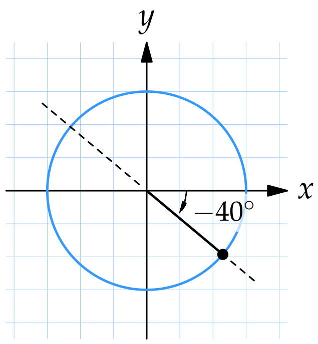
Since the angle $209°$ can be written as $209° = 180° + 29°$ the angle corresponds to a point on the unit circle which lies in the third quadrant. The point has a negative x-coordinate, which means that $\cos 209°$ is negative.

b) $\sin 133°$



The angle $133°$ is equal to $90° + 43°$ and gives a point on the unit circle which lies in the second quadrant. The quadrant has points with positive y-coordinate, and therefore $\sin 133°$ is positive.

c) $\tan(-40°)$



By drawing angle $-40°$ in the unit circle one obtains a radial line which has a negative slope, so that $\tan(-40°)$ is negative.

Example 9

Calculate $\sin \dfrac{2\pi}{3}$.

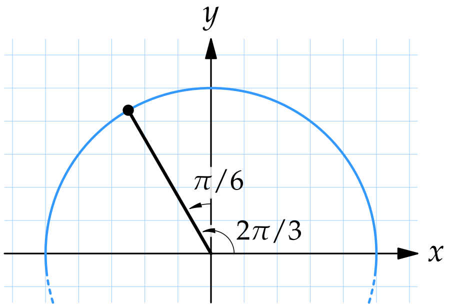
Note that

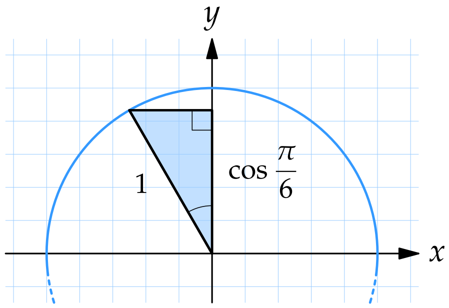
$$\frac{2\pi}{3} = \frac{4\pi}{6} = \frac{3\pi + \pi}{6} = \frac{\pi}{2} + \frac{\pi}{6}.$$

This shows that the point on the unit circle corresponding to the angle $2\pi/3$ is in the the second quadrant and makes the angle $\pi/6$ with the positive y-axis. If we draw an extra triangle as in the figure below on the right, we see that the $2\pi/3$- point on the unit circle has a y-coordinate which is equal to the adjacent side

$\cos\frac{\pi}{6} = \sqrt{3}/2$. So we have that

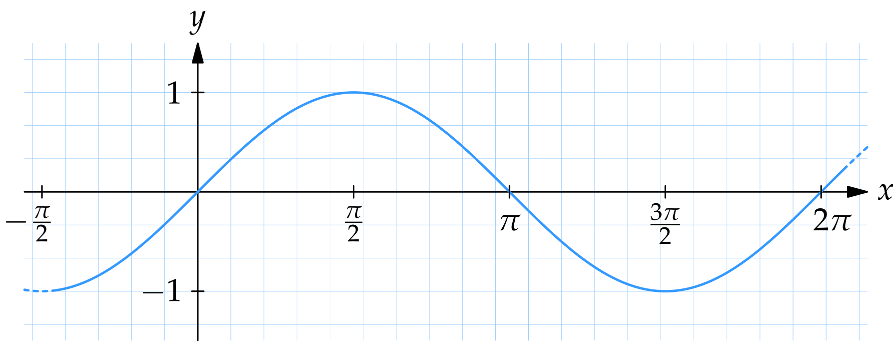
$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}.$$



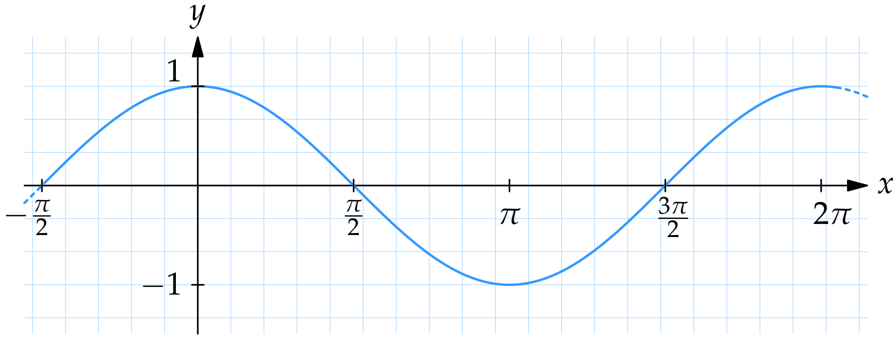


Graphs of the trigonometric functions

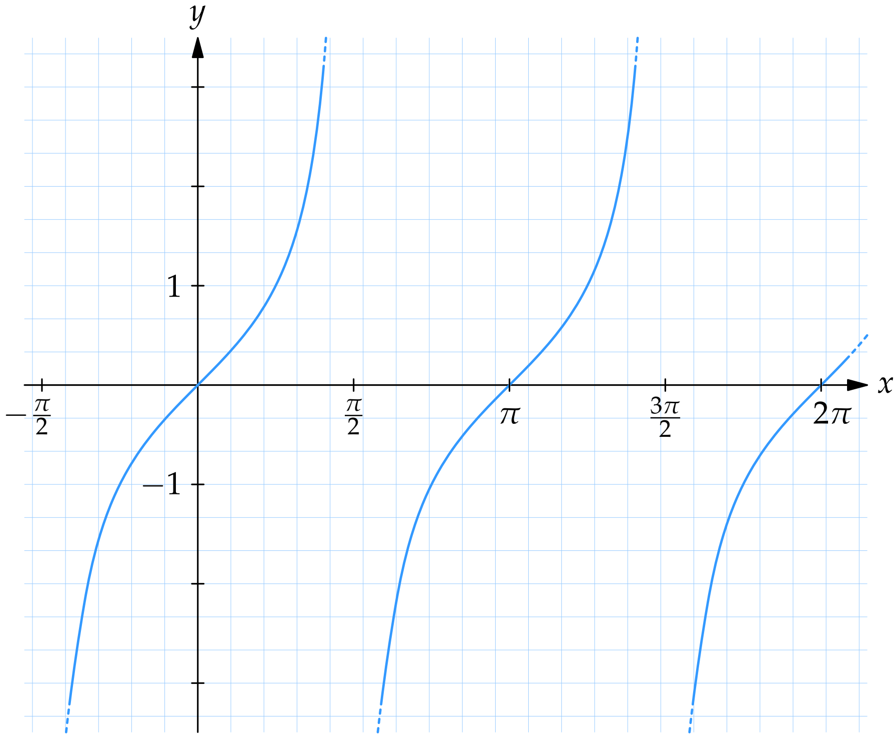
In the last section we used a unit circle to define the cosine and sine of arbitrary angles, and we will often use the unit circle in the future, for example, to derive trigonometric relationships and solve trigonometric equations. However, there are certain characteristics of the trigonometric functions that are better illustrated by drawing their graphs.



The graph of the sine function.



The graph of the cosine function.



The graph of the tangent function.

In these graphs, we might observe several things more clearly than in the unit circle. Some examples are:

- The curves for cosine and sine repeat themselves after a change in angle of 2π, that is $\cos(x + 2\pi) = \cos x$ and $\sin(x + 2\pi) = \sin x$. To see why this is true, note that on the unit circle 2π corresponds to a complete revolution, and after a complete revolution we return to the same point on the circle.

- The curve for the tangent repeats itself after a change in angle of π, that is $\tan(x + \pi) = \tan x$. Two angles which differ by π share the same line through the origin of the unit circle and thus their radial lines have the same slope.

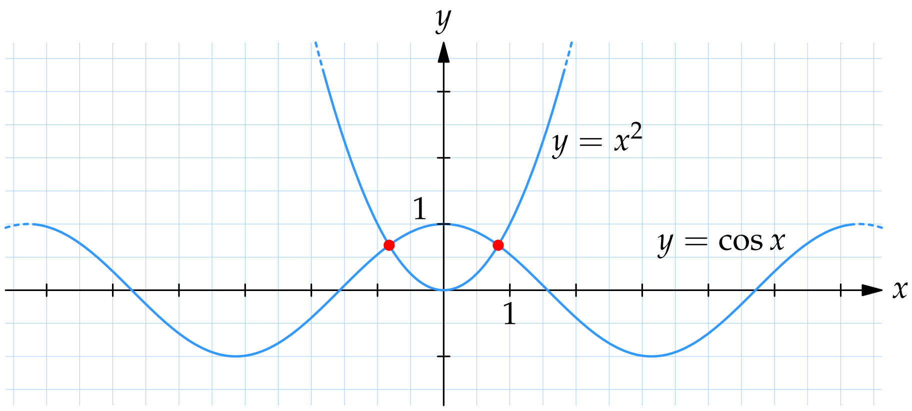
- Except for a phase shift of $\pi/2$ the curves for cosine and sine are identical, that is $\cos x = \sin(x + \pi/2)$; more about this in the next section.

The curves can also be important when examining trigonometric equations. With a simple sketch, you can often get an idea of how many solutions an equation has, and where the solutions lie.

Example 10

How many solutions has the equation $\cos x = x^2$ (where x is measured in radians)?

By drawing the graphs $y = \cos x$ and $y = x^2$ we see that the curves intersect in two points. So there are two x-values for which the corresponding y-values are equal. In other words, the equation has two solutions.



Study advice

Basic and final tests

After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that...

If you have studied trigonometry, then you should not be afraid to use it in geometric problems. It often produces a simpler solution.

You may need to spend a lot of time understanding how to use a unit circle to define the trigonometric functions.

You should get into the habit of calculating with precise trigonometric values. It is good training in calculating fractions and will eventually help you handle algebraic rational expressions.

Reviews
For those of you who want to deepen your understanding or need more detailed explanations consider the following references:

- Learn more about trigonometry from Wikipedia (`http://en.wikipedia.org/wiki/Trigonometric_function`)
- Learn more about the unit circle from Wikipedia (`http://en.wikipedia.org/wiki/Unit_circle`)

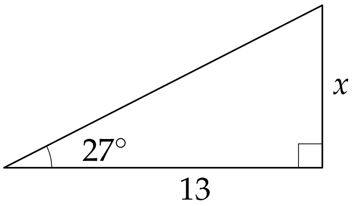
Useful web sites

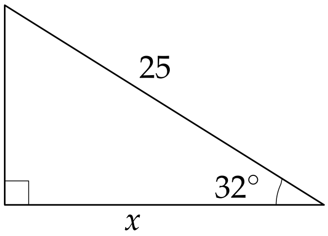
- Experiment with the sine and cosine functions (`http://www.math.kth.se/online/images/sinus_och_cosinus_i_enhetscirkeln.swf`)
- Experiment with Euclidean geometry (`http://www.math.psu.edu/dlittle/java/geometry/euclidean/toolbox.html`)

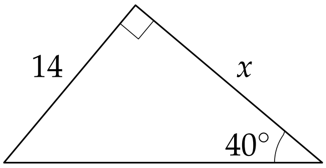
4.2 Exercises

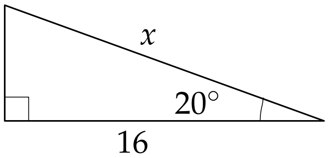
Exercise 4.2:1

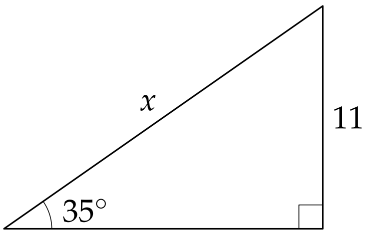
Using the trigonometric functions, determine the length of the side marked x.

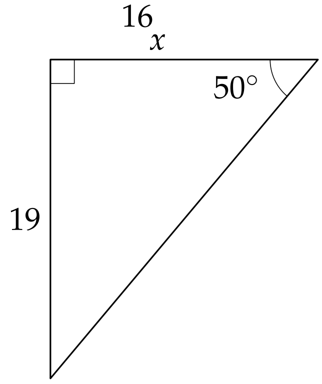
a) 

b) 

c) 

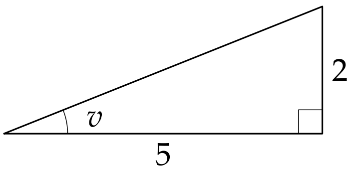
d) 

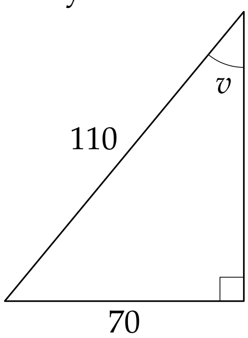
e) 

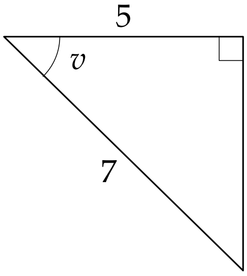
f) 

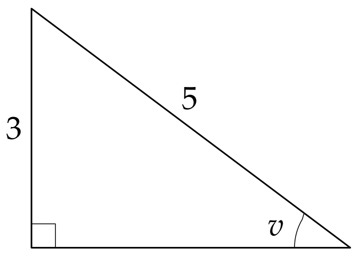
Exercise 4.2:2

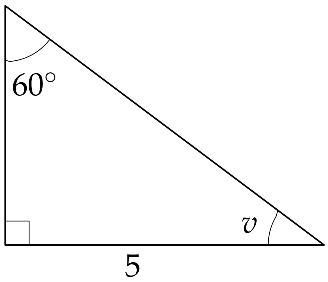
Determine a trigonometric equation that is satisfied by v.

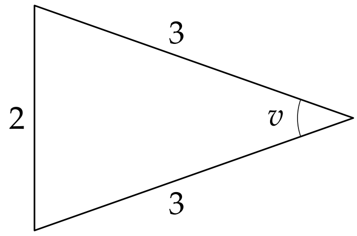
a) 

b) 

c) 

d) 

e) 

f) 

Exercise 4.2:3

Determine

a)	$\sin\left(-\frac{\pi}{2}\right)$	b)	$\cos 2\pi$	c)	$\sin 9\pi$
d)	$\cos\frac{7\pi}{2}$	e)	$\sin\frac{3\pi}{4}$	f)	$\cos\left(-\frac{\pi}{6}\right)$

Exercise 4.2:4

Determine

a)	$\cos\frac{11\pi}{6}$	b)	$\cos\frac{11\pi}{3}$	c)	$\tan\frac{3\pi}{4}$
d)	$\tan\pi$	e)	$\tan\frac{7\pi}{6}$	f)	$\tan\left(-\frac{5\pi}{3}\right)$

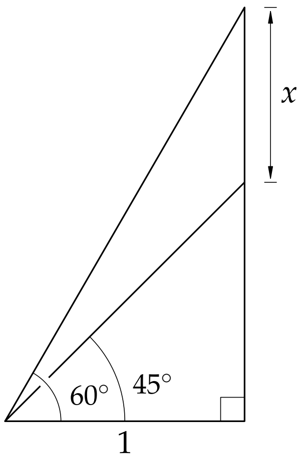
Exercise 4.2:5

Determine

a)	$\cos 135^\circ$	b)	$\tan 225^\circ$	c)	$\cos 330^\circ$	d)	$\tan 495^\circ$

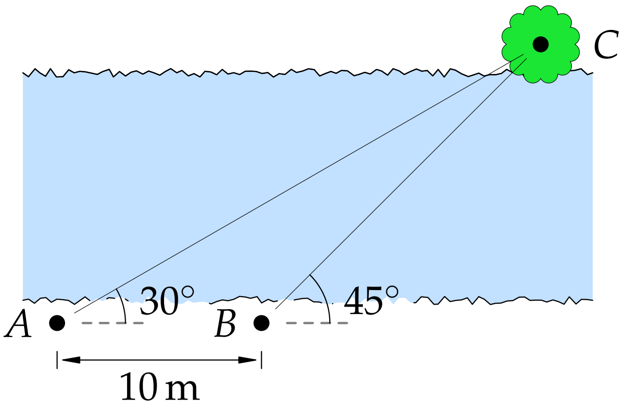
Exercise 4.2:6

Determine the length of the side marked x.



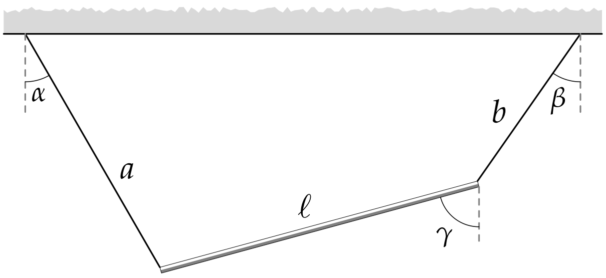
Exercise 4.2:7

In order to determine the width of a river, we measure from two points, A and B on one side of the straight bank to a tree, C, on the opposite side. How wide is the river if the measurements in the figure are correct?



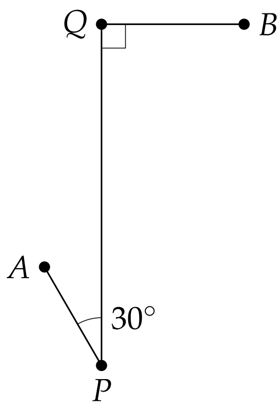
Exercise 4.2:8

A rod of length ℓ hangs from two ropes of length a and b as shown in the figure. The ropes make angles α and β with the vertical. Determine a trigonometric equation for the angle γ which the rod makes with the vertical.



Exercise 4.2:9

The road from A to B consists of three straight parts AP, PQ and QB, which are 4.0 km, 12.0 km and 5.0 km respectively. The angles marked at P and Q in the figure are 30° and 90° respectively. Calculate the distance as the crow flies from A to B. (The exercise is taken from the Swedish National Exam in Mathematics, November 1976, although slightly modified.)



4.3 Trigonometric relations

Contents:

- The Pythagorean identity
- The double-angle and half-angle formulas
- Addition and subtraction formulas

Learning outcome:

After this section, you will have learned how to:

- Derive trigonometric relationships from symmetries in the unit circle.
- Simplify trigonometric expressions with the help of trigonometric formulas.

Introduction

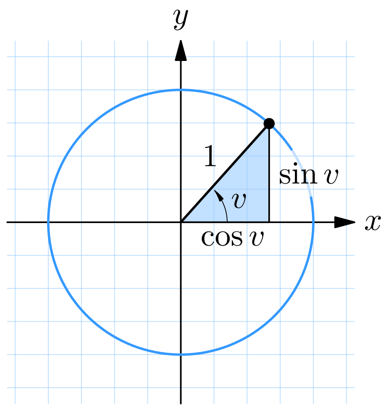
There are a variety of trigonometric formulas which relate the sine, cosine or tangent for an angle or multiples of an angle. They are usually known as the trigonometric identities. Here we will give some of these trigonometric relationships, and show how to derive them. There are many more than we can deal with in this course, but most can be derived from the so-called Pythagorean identity and the addition and subtraction formulas (see below), which are important to know by heart.

The Pythagorean identity

This identity is the most basic, but is in fact nothing more than Pythagorean theorem, applied to the unit circle. The right-angled triangle on the right shows that

$$(\sin v)^2 + (\cos v)^2 = 1,$$

which is usually written as $\sin^2 v + \cos^2 v = 1$.



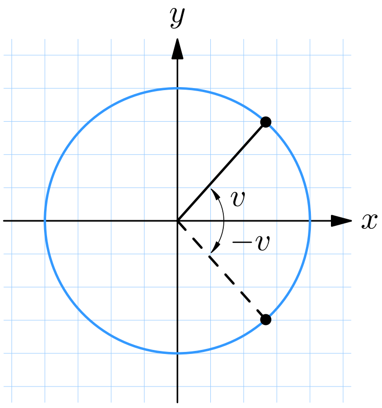
Symmetries

With the help of the unit circle and by exploiting the symmetries we obtain a large number of relationships between the cosine and sine functions:

$$\begin{aligned} \cos(-v) &= \cos v & \cos\left(\frac{\pi}{2} - v\right) &= \sin v \\ \sin(-v) &= -\sin v & \sin\left(\frac{\pi}{2} - v\right) &= \cos v \\ \cos(\pi - v) &= -\cos v & \cos\left(v + \frac{\pi}{2}\right) &= -\sin v \\ \sin(\pi - v) &= \sin v & \sin\left(v + \frac{\pi}{2}\right) &= \cos v \end{aligned}$$

Instead of trying to learn all of these relationships by heart, it is better to learn how to derive them from the unit circle.

Reflection in the *x*-axis

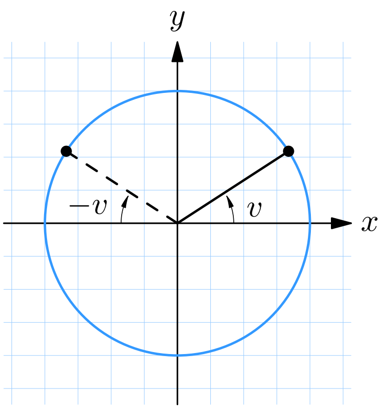


When an angle v is reflected in the x-axis it becomes $-v$.

Reflection does not affect the x-coordinate while the y-coordinate changes sign, so that

$$\begin{aligned} \cos(-v) &= \cos v, \\ \sin(-v) &= -\sin v. \end{aligned}$$

Reflection in the *y*-axis

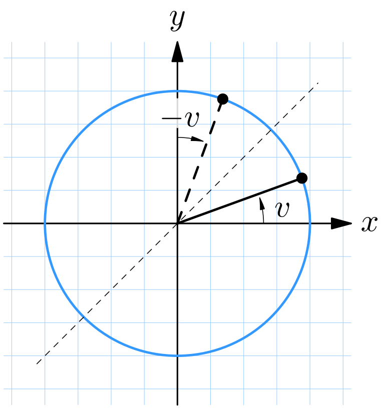


Reflection in the y-axis changes the angle from v to $\pi - v$ (the reflection makes an angle v with the negative x-axis).

This reflection does not affect the y-coordinate, while the x-coordinate changes sign, so we see that

$$\begin{aligned} \cos(\pi - v) &= -\cos v, \\ \sin(\pi - v) &= \sin v. \end{aligned}$$

Reflection in the line $y = x$

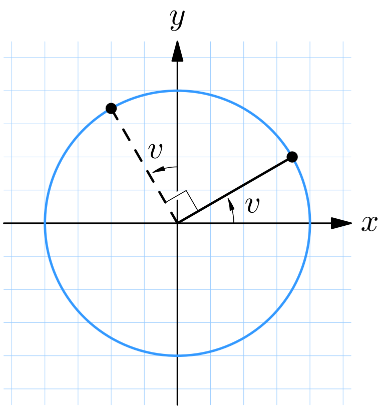


The angle v is changed to $\pi/2 - v$ (the reflected line makes an angle v with the positive y-axis).

This reflection swaps the x- and y-coordinates, so that

$$\cos\left(\frac{\pi}{2} - v\right) = \sin v,$$
$$\sin\left(\frac{\pi}{2} - v\right) = \cos v.$$

Rotation by an angle of $\pi/2$



A rotation $\pi/2$ of the angle v means that the angle becomes $v + \pi/2$.

The rotation turns the x-coordinate into the new y-coordinate and the y-coordinate turns into the new x-coordinate with the opposite sign, so that

$$\cos\left(v + \frac{\pi}{2}\right) = -\sin v,$$
$$\sin\left(v + \frac{\pi}{2}\right) = \cos v.$$

The addition, subtraction and double-angle formulas

We often need to deal with expressions in which two or more angles are involved, such as $\sin(u + v)$. We will then need the so-called “addition formulas” . For sine and cosine the formulas are:

$$\begin{aligned}
\sin(u+v) &= \sin u\, \cos v + \cos u\, \sin v,\\
\sin(u-v) &= \sin u\, \cos v - \cos u\, \sin v,\\
\cos(u+v) &= \cos u\, \cos v - \sin u\, \sin v,\\
\cos(u-v) &= \cos u\, \cos v + \sin u\, \sin v.
\end{aligned}$$

If we want to know the sine or cosine of a double angle, that is $\sin 2v$ or $\cos 2v$, we can use the addition formulas above to get the double-angle formulas:

$$\sin 2v = 2\sin v \cos v,$$
$$\cos 2v = \cos^2 v - \sin^2 v.$$

From these relationships, we can also derive formulas for half angles. By replacing $2v$ by v, and consequently v by $v/2$, in the formula for $\cos 2v$ we get that

$$\cos v = \cos^2\frac{v}{2} - \sin^2\frac{v}{2}.$$

If we want a formula for $\sin(v/2)$ we use the Pythagorean identity to write $\cos^2(v/2)$ in terms of $\sin^2(v/2)$. Then

$$\cos v = 1 - \sin^2\frac{v}{2} - \sin^2\frac{v}{2} = 1 - 2\sin^2\frac{v}{2}$$

so that

$$\sin^2\frac{v}{2} = \frac{1-\cos v}{2}.$$

Similarly, we can write $\sin^2(v/2)$ in terms of $\cos^2(v/2)$. Then we will arrive at

$$\cos^2\frac{v}{2} = \frac{1+\cos v}{2}.$$

Study advice

The basic and final tests

After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that…

The unit circle is an invaluable tool for deriving trigonometric relationships. There are lots of these, and there is no point in trying to learn all of them by heart. It is also time-consuming to have to look them up all the time. Therefore, it is much better that you learn how to use the unit circle.

The most famous trigonometric formula is the so-called Pythagorean identity. It applies to all angles, not just acute angles. It is based on the Pythagoras theorem.

Useful web sites

- Experiment with the cosine "box"
(`http://www.ies.co.jp/math/java/trig/cosbox/cosbox.html`)

4.3 Exercises

Exercise 4.3:1

Determine the angles v between $\dfrac{\pi}{2}$ and 2π which satisfy

a) $\cos v = \cos \dfrac{\pi}{5}$ b) $\sin v = \sin \dfrac{\pi}{7}$ c) $\tan v = \tan \dfrac{2\pi}{7}$

Exercise 4.3:2

Determine the angles v between 0 and π which satisfy

a) $\cos v = \cos \dfrac{3\pi}{2}$ b) $\cos v = \cos \dfrac{7\pi}{5}$

Exercise 4.3:3

Suppose that $-\dfrac{\pi}{2} \le v \le \dfrac{\pi}{2}$ and that $\sin v = a$. With the help of a express

a) $\sin(-v)$ b) $\sin(\pi - v)$

c) $\cos v$ d) $\sin\left(\dfrac{\pi}{2} - v\right)$

e) $\cos\left(\dfrac{\pi}{2} + v\right)$ f) $\sin\left(\dfrac{\pi}{3} + v\right)$

Exercise 4.3:4

Suppose that $0 \le v \le \pi$ and that $\cos v = b$. With the help of b express

a) $\sin^2 v$ b) $\sin v$

c) $\sin 2v$ d) $\cos 2v$

e) $\sin\left(v + \dfrac{\pi}{4}\right)$ f) $\cos\left(v - \dfrac{\pi}{3}\right)$

Exercise 4.3:5

Determine $\cos v$ and $\tan v$, where v is an acute angle in a triangle such that $\sin v = \dfrac{5}{7}$.

Exercise 4.3:6

a) Determine $\sin v$ and $\tan v$ if $\cos v = \dfrac{3}{4}$ and $\dfrac{3\pi}{2} \le v \le 2\pi$.

b) Determine $\cos v$ and $\tan v$ if $\sin v = \dfrac{3}{10}$ and v lies in the second quadrant.

c) Determine $\sin v$ and $\cos v$ if $\tan v = 3$ and $\pi \le v \le \dfrac{3\pi}{2}$.

Exercise 4.3:7

Determine $\sin(x+y)$ if

a) $\sin x = \dfrac{2}{3}$, $\sin y = \dfrac{1}{3}$ and x, y are angles in the first quadrant.

b) $\cos x = \dfrac{2}{5}$, $\cos y = \dfrac{3}{5}$ and x, y are angles in the first quadrant.

Exercise 4.3:8

Show the following trigonometric relations

a) $\tan^2 v = \dfrac{\sin^2 v}{1 - \sin^2 v}$

b) $\dfrac{1}{\cos v} - \tan v = \dfrac{\cos v}{1 + \sin v}$

c) $\tan \dfrac{u}{2} = \dfrac{\sin u}{1 + \cos u}$

d) $\dfrac{\cos(u + v)}{\cos u \cos v} = 1 - \tan u \tan v$

Exercise 4.3:9

Show *Morries equality*:

$$\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ = \frac{1}{8}.$$

(Hint: use the formula for double angles on $\sin 160^\circ$.)

4.4 Trigonometric equations

Contents:

- Basic trigonometric equations
- Simple trigonometric equations

Learning outcomes:

After this section, you will have learned how to:

- Solve the basic equations of trigonometry.
- Solve trigonometric equations that can be reduced to basic equations.

Basic equations

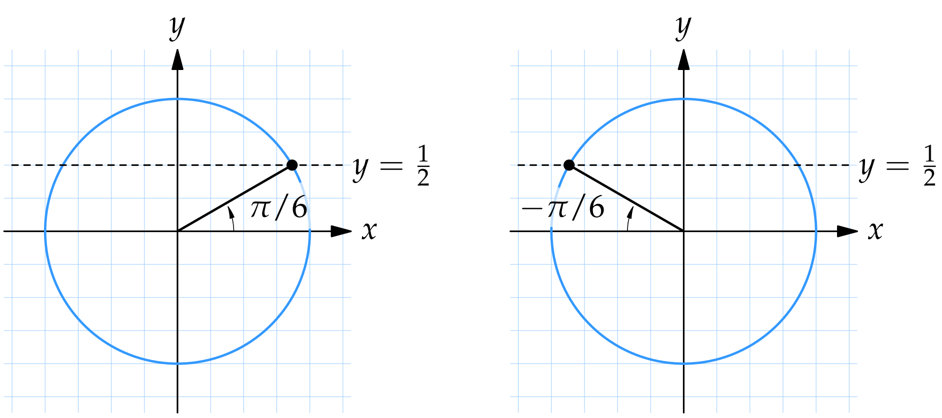
Trigonometric equations can be very complicated, but there are also many types which can be solved using relatively simple methods. Here we shall start by looking at the most basic trigonometric equations, of the type $\sin x = a$, $\cos x = a$ and $\tan x = a$.

These equations usually have an infinite number of solutions, unless the circumstances limit the number of possible solutions (for example, if one is looking for an acute angle).

Example 1

Solve the equation $\sin x = \frac{1}{2}$.

Our task is to determine all the angles that have a sine equal to $\frac{1}{2}$. The unit circle helps us in this. Note that here the angle is designated as x.



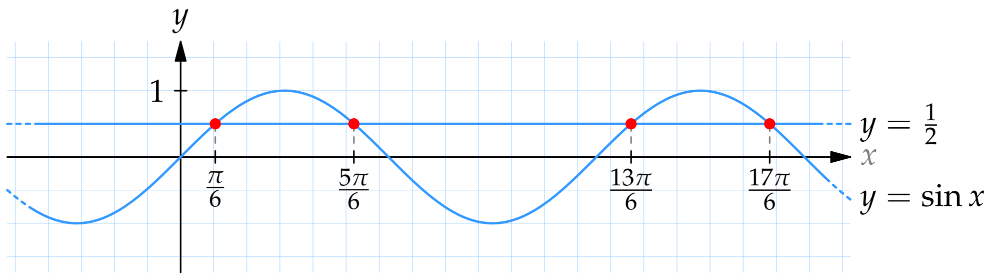
The figure illustrates the two points on the circle which have y-coordinate $\frac{1}{2}$, i.e. the two points whose corresponding angles have sine value $\frac{1}{2}$. The first is the standard angle $30^\circ = \pi/6$ and by symmetry the other angle makes 30° with the negative x-axis. Therefore the other angle is 180°˘$30^\circ = 150^\circ$ or in radians π˘$\pi/6 = 5\pi/6$. These are the only solutions to the equation $\sin x = \frac{1}{2}$ between 0 and 2π.

However, we can add an arbitrary number of revolutions to these two angles and still get the same value for the sine . Thus all angles x for which $\sin x = \frac{1}{2}$ are

$$\begin{cases} x = \dfrac{\pi}{6} + 2n\pi, \\[2ex] x = \dfrac{5\pi}{6} + 2n\pi, \end{cases}$$

where n is an arbitrary integer. This is called the general solution to the equation.

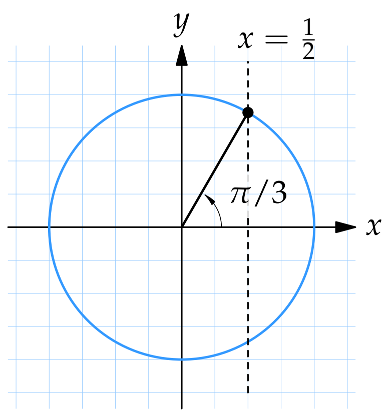
The solutions can also be obtained from the figure below, by looking at where the graph of $y = \sin x$ intersects the line $y = \frac{1}{2}$.

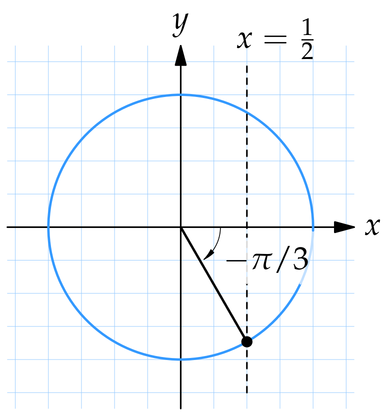


Example 2

Solve the equation $\cos x = \frac{1}{2}$.

We once again study the unit circle.





We know that cosine is $\frac{1}{2}$ for the angle $\pi/3$. The only other point on the unit circle which has x-coordinate $\frac{1}{2}$ corresponds to the angle $-\pi/3$. Adding an integral number of revolutions to these angles we get the general solution

$$x = \pm\pi/3 + n \cdot 2\pi,$$

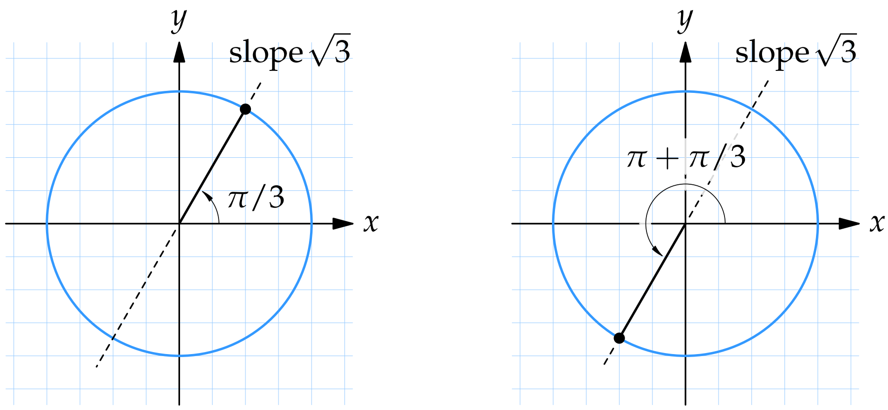
where n is an arbitrary integer.

Example 3

Solve the equation $\tan x = \sqrt{3}$.

One solution to the equation is the standard angle $x = \pi/3$.

If we study the unit circle then we see that the tangent of an angle is equal to the slope of the straight line through the origin which makes an angle x with the positive x-axis .



Therefore, we see that the solutions to $\tan x = \sqrt{3}$ repeat themselves every half revolution, and so the general solution can be obtained from the solution $\pi/3$ by adding or subtracting multiples of π:

$$x = \pi/3 + n \cdot \pi,$$

where n is an arbitrary integer.

Somewhat more complicated equations

Trigonometric equations can vary in many ways, and it is impossible to give a full catalogue of all possible equations. But let us study some examples where we can use our knowledge of solving basic equations.

Some trigonometric equations can be simplified by rewriting them with the help of the trigonometric relationships. This, for example, could lead to a quadratic equation, as in the example below.

Example 4

Solve the equation $\cos 2x - 4\cos x + 3 = 0$.

Rewrite by using the formula $\cos 2x = 2\cos^2 x - 1$, so that

$$(2\cos^2 x - 1) - 4\cos x + 3 = 0.$$

Simplifying, we see that

$$\cos^2 x - 2\cos x + 1 = 0.$$

The left-hand side can factorised by using the squaring rule to give

$$(\cos x - 1)^2 = 0.$$

This equation can only be satisfied if $\cos x = 1$. The basic equation $\cos x = 1$ can be solved in the normal way, and thus the complete solution is

$$x = 2n\pi. \qquad (n \text{ arbitrary integer}).$$

Example 5

Solve the equation $\frac{1}{2}\sin x + 1 - \cos^2 x = 0$.

According to the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, i.e. $1 - \cos^2 x = \sin^2 x$. Thus the equation can be written as

$$\tfrac{1}{2}\sin x + \sin^2 x = 0.$$

Factorising out $\sin x$ we get

$$\sin x \cdot \left(\tfrac{1}{2} + \sin x\right) = 0.$$

From this factorised form of the equation, we see that the solutions either have to satisfy $\sin x = 0$ or $\sin x = -\frac{1}{2}$, which are two basic equations of the type $\sin x = a$ and can be solved as in Example 1. The solutions turn out to be

$$\begin{cases} x &= n\pi, \\ x &= -\pi/6 + 2n\pi, \\ x &= 7\pi/6 + 2n\pi. \end{cases} \qquad (n \text{ arbitrary integer}).$$

Example 6

Solve the equation $\sin 2x = 4\cos x$.

By rewriting the equation using the formula for double-angles we get

$$2\sin x\ \cos x - 4\cos x = 0.$$

Dividing both sides by 2 and factorising out $\cos x$, gives

$$\cos x\ \cdot\ (\sin x - 2) = 0.$$

The left-hand side can only be zero if one of the factors is zero, and we have reduced the original equation into two basic equations:

- $\cos x = 0$,
- $\sin x = 2$.

However, $\sin x$ can never be greater than 1, so the equation $\sin x = 2$ has no solutions. That just leaves $\cos x = 0$, and using the unit circle we see that the general solution is

$$x = \frac{\pi}{2} + n\pi,$$

for n an arbitrary integer.

Example 7

Solve the equation $4\sin^2 x - 4\cos x = 1$.

Using the Pythagorean identity we can replace $\sin^2 x$ by $1 - \cos^2 x$. Then

$$\begin{aligned} 4(1-\cos^2 x) - 4\cos x &= 1, \\ 4 - 4\cos^2 x - 4\cos x &= 1, \\ -4\cos^2 x - 4\cos x + 4 - 1 &= 0, \\ \cos^2 x + \cos x - \tfrac{3}{4} &= 0. \end{aligned}$$

This is a quadratic equation in $\cos x$, which has the solutions

$$\cos x = -\tfrac{3}{2} \quad \text{and} \quad \cos x = \tfrac{1}{2}.$$

Since the value of $\cos x$ is between -1 and 1, the equation $\cos x = -\frac{3}{2}$ has no solutions. That leaves only the basic equation

$$\cos x = \tfrac{1}{2},$$

which may be solved as in Example 2.

Study advice

Basic and final tests
After you have read the text and worked through the exercises, you should do the basic and final tests to pass this section. You can find the link to the tests in your student lounge.

Keep in mind that . . .
It is a good idea to learn the most common trigonometric formulas (identities) and practice simplifying and manipulating trigonometric expressions.

It is important to be familiar with the basic equations, such as $\sin x = a$, $\cos x = a$ or $\tan x = a$ (where a is a real number). It is also important to know that these equations typically have infinitely many solutions.

Useful web sites

- Experiment with the graph $y = a \sin[b(x - c)]$.
(`http://www.ies.co.jp/math/java/trig/ABCsinX/ABCsinX.html`)

4.4 Exercises

Exercise 4.4:1

For which angles v, where $0 \le v \le 2\pi$, does

a) $\sin v = \dfrac{1}{2}$

b) $\cos v = \dfrac{1}{2}$

c) $\sin v = 1$

d) $\tan v = 1$

e) $\cos v = 2$

f) $\sin v = -\dfrac{1}{2}$

g) $\tan v = -\dfrac{1}{\sqrt{3}}$

Exercise 4.4:2

Solve the equation

a) $\sin x = \dfrac{\sqrt{3}}{2}$

b) $\cos x = \dfrac{1}{2}$

c) $\sin x = 0$

d) $\sin 5x = \dfrac{1}{\sqrt{2}}$

e) $\sin 5x = \dfrac{1}{2}$

f) $\cos 3x = -\dfrac{1}{\sqrt{2}}$

Exercise 4.4:3

Solve the equation

a) $\cos x = \cos \dfrac{\pi}{6}$

b) $\sin x = \sin \dfrac{\pi}{5}$

c) $\sin\,(x + 40^\circ) = \sin 65^\circ$

d) $\sin 3x = \sin 15^\circ$

Exercise 4.4:4

Determine the angles v in the interval $0^\circ \le v \le 360^\circ$ which satisfy $\cos\,(2v + 10^\circ) = \cos 110^\circ$.

Exercise 4.4:5

Solve the equation

a) $\sin 3x = \sin x$

b) $\tan x = \tan 4x$

c) $\cos 5x = \cos(x + \pi/5)$

Exercise 4.4:6

Solve the equation

a) $\sin x \cdot \cos 3x = 2 \sin x$

b) $\sqrt{2} \sin x \cos x = \cos x$

c) $\sin 2x = -\sin x$

Exercise 4.4:7

Solve the equation

a) $2\sin^2 x + \sin x = 1$

b) $2\sin^2 x - 3\cos x = 0$

c) $\cos 3x = \sin 4x$

Exercise 4.4:8

Solve the equation

a) $\sin 2x = \sqrt{2}\cos x$

b) $\sin x = \sqrt{3}\cos x$

c) $\dfrac{1}{\cos^2 x} = 1 - \tan x$

5.1 Writing formulas in TEX

Contens:

- LATEX maths

Learning outcomes:

After this section you will have learned how to:

- Write simple maths formulas in LATEX.
- Avoid common mistakes when coding maths in LATEX.

To write mathematics efficiently on a computer in your individual assignment and the group task you will need to write the maths in a coded form called LATEX syntax. In this section you will learn the fundamentals of constructing LATEX code that yields simple maths formulas.

How to write basic expressions

To indicate the **start** of math formatting, use the tag `<math>`. To **end** math formatting, use the tag `</math>`. For example, if you want the formula $a+b$, in the text box write `<math>a+b</math>`.

Simple mathematical formulae are written in a straightforward way.

Example 1

a) $1+2-3$ is written `<math>1+2-3</math>`

b) $5/2$ is written `<math>5/2</math>`

c) $4/(2+x)$ is written `<math>4/(2+x)</math>`

d) $4<5$ is written `<math>4 < 5</math>`

When you need to use symbols that are not available on the keyboard or construct formulas that are not simple you use special commands that start with a backslash, e.g. `\le` is a command that gives you $\leq$.

The table below shows some of the most commonly used maths commands in LaTeX.

	Example	LaTeX	Comment
Simple operations	$a+b$	`a+b`	
	$a-b$	`a-b`	
	$a\pm b$	`a\pm b`	
	$a\cdot b$	`a\cdot b`	
	a/b	`a/b`	
	$\frac{1}{2}$	`\frac{1}{2}`	Small stacked fraction
	$\dfrac{a}{b}$	`\dfrac{a}{b}`	Large stacked fraction
	(a)	`(a)`	Scalable parantheses: `\left(...\right)`
Relation signs	$a=b$	`a=b`	
	$a\ne b$	`a\ne b`	Alternatively: `a\not=b`
	$a< b$	`a< b`	NB: Space after “<”
	$a\le b$	`a\le b`	
	$a>b$	`a>b`	
	$a\ge b$	`a\ge b`	
Powers and roots	x^{n}	`x^{n}`	
	$\sqrt{x}$	`\sqrt{x}`	
	$\sqrt[n]{x}$	`\sqrt[n]{x}`	
Index	x_{n}	`x_{n}`	
Logarithms	$\lg x$	`\lg x`	
	$\ln x$	`\ln x`	
	$\log x$	`\log x`	
	$\log_{a} x$	`\log_{a} x`	
Trigonometry	30°	`30^{\circ}`	
	$\cos x$	`\cos x`	

	$\sin x$	`\sin x`	
	$\tan x$	`\tan x`	
	$\cot x$	`\cot x`	
Arrows	$\Rightarrow$	`\Rightarrow`	Implies
	$\Leftarrow$	`\Leftarrow`	Is implied by
	$\Leftrightarrow$	`\Leftrightarrow`	Is equivalent to
Verschiedene Symbole	π	`\pi`	
	$\alpha, \beta, \theta, \varphi$	`\alpha, \beta, \theta, \varphi`	

Example 2

a) $1 \pm 3 \cdot 5$ is written `<math>1\pm 3\cdot 5</math>`

b) $\frac{1}{2}y \ne x \le z$ is written `<math>\frac{1}{2}y\ne x\le z</math>`

c) $2^{13}\sqrt{3} + \ln y$ is written `<math>2^{13}\sqrt{3}+\ln y</math>`

d) $\tan 30^{\circ}$ is written `<math>\tan 30^{\circ}</math>`

How to write complex expressions

By combining simple expressions, we may form more complex expressions.

Example 3

a) $\sqrt{x+2}$ is written `<math>\sqrt{x+2}</math>`

b) $(a^2)^3 = a^6$ is written `<math>(a^2)^3=a^6</math>`

c) $2^{(2^2)}$ is written `<math>2^{(2^2)}</math>`

d) $\sin\sqrt{x}$ is written `<math>\sin\sqrt{x}</math>`

Example 4

a) $\sqrt{x+\sqrt{x}}$ is written $\sqrt{x+\sqrt{x}}$

b) $\dfrac{x-x^2}{\sqrt{3}}$ is written $\dfrac{x-x^2}{\sqrt{3}}$

c) $\dfrac{x}{x+\dfrac{1}{x}}$ is written $\dfrac{x}{x+\dfrac{1}{x}}$

d) $x_{1,2}=-\dfrac{p}{2}\pm\sqrt{\left(\dfrac{p}{2}\right)^2-q}$ is written $x_{1,2}= -\dfrac{p}{2}\pm\sqrt{\left(\dfrac{p}{2}\right)^2-q}$

How to avoid common mistakes

One of the most common mistakes when editing math in the wiki is to forget the start $tag and the end$ tag.

Remember also to start commands with a backslash (\) and to add a space after the commands (unless they are followed immediately by a new command).

Another frequent mistake is to use an asterisk (*) instead of a proper multiplication sign $\cdot$ (\cdot in TEX).

Example 5

		LATEX	Result
a)	Don't forget backslash (\)	sin x	$sinx$
	Remember to put a space after a command	\sinx	Error
	Do write	\sin x	$\sin x$
b)	Don't use asterisk as a multiplication sign	4*3	$4 * 3$
	Do write	4\cdot 3	$4 \cdot 3$

c)	The multiplication sign is usually not inserted between variables	`a\cdot b`	$a \cdot b$
	Do write	`ab`	ab

Superscripts and subscripts

When writing superscripts, such as exponents, you use `^`, and to write subscripts you use `_`. If the super- or subscript consists of more than one symbol it must be enclosed with braces `{}`.

A special kind of superscript is the degree sign ($^\circ$) which is written as `^{\circ}`.

Example 6

		LaTeX	Result
a)	Don't omit ^	`a2`	$a2$
	Do write	`a^2`	a^2
b)	Don't omit _	`x1`	$x1$
	Do write	`x_1`	x_1
c)	Remember to use braces	`a^22`	a^22
	Do write	`a^{22}`	a^{22}
d)	Don't use "o" as the degree sign	`30^{o}`	30^{o}
	Don't use "0" as the degree sign	`30^{0}`	30^{0}
	Do write	`30^{\circ}`	30°

Delimiters

In more complex expressions you need to make sure to balance each opening parenthesis `(` with a closing parenthesis `)`.

A pair of parenthesis that delimits a tall expression should be as large as the expression. You should therefore prefix the opening parenthesis with `\left` and the closing parenthesis with `\right` to get a pair of extensible parentheses that adjust its height to the expression.

Note also that braces, `{}`, and not parentheses, `()`, are used to delimit arguments (input values) of commands such as `\sqrt` and `\frac`.

Example 7

		LaTeX	Result
a)	Use the correct number of brackets	`(1-(1-x)`	$(1-(1-x)$
	Do write	`(1-(1-x))`	$(1-(1-x))$
b)	Brackets should be as large as the expression	`(\dfrac{a}{b}+c)`	$(\dfrac{a}{b}+c)$
	Do write	`\left(\dfrac{a}{b}+c\right)`	$\left(\dfrac{a}{b}+c\right)$
c)	Don't use parentheses to delimit arguments	`\frac(1)(2)`	$\frac(1)(2)$
	Do write	`\frac{1}{2}`	$\frac{1}{2}$
d)	Don't use parentheses to delimit arguments	`\sqrt(a+b)`	$\sqrt(a+b)$
	Avoid redundant brackets	`\sqrt{(a+b)}`	$\sqrt{(a+b)}$
	Do write	`\sqrt{a+b}`	$\sqrt{a+b}$

Fractions

As a rule of thumb you should write fractions where the numerator and denominator consist only of a few digits as a small fraction (i.e. with `\frac`), while other fractions should be large (i.e. with `\dfrac`).

If an exponent or index contains a fraction then that fraction should be written in a slashed form (e.g. $5/2$ instead of $\frac{5}{2}$) to enhance the legibility.

Example 8

		LaTeX	Result
a)	Don't write numerical fractions large	`\dfrac{1}{2}`	$\dfrac{1}{2}$
	Do write	`\frac{1}{2}`	$\frac{1}{2}$
	(Exception: If the fraction is next to a large expression you should, however, write the fraction as a large fraction.)		
b)	Don't write symbolic fractions small	`\frac{a}{b}`	$\frac{a}{b}$
	Do write	`\dfrac{a}{b}`	$\dfrac{a}{b}$
c)	Don't write complicated fractions small	`\frac{\sqrt{3}}{2}`	$\frac{\sqrt{3}}{2}$
	Do write	`\dfrac{\sqrt{3}}{2}`	$\dfrac{\sqrt{3}}{2}$
d)	No stacked fractions in exponents	`a^{\frac{1}{2}}`	$a^{\frac{1}{2}}$
	Do write	`a^{1/2}`	$a^{1/2}$

Study advice

A tip is to try out your maths formulas in the forum or in the wiki where you work on your individual assignment.

Useful web sites

- A more thorough list of LaTeX maths commands can be found on Wikipedia's help page (`http://en.wikipedia.org/wiki/Help:Displaying_a_formula`)

- Two more thorough texts on LaTeX maths can be found in a chapter of the "The LaTeX Companion" (`http://www.cism.it/cism/volconts/ch8.pdfachapter`) and a text by Herbert Voss (`http://www.tex.ac.uk/tex-archive/info/math/voss/mathmode/Mathmode.pdftext`)
- If you want to know more about LaTeX you can visit these sites: Wikipedia (`http://en.wikipedia.org/wiki/LaTeX`), The not so Short Introduction to LaTeX (`http://www.ctan.org/tex-archive/info/lshort/english/lshort.pdf`) and LaTeX Wikibook (`http://en.wikibooks.org/wiki/LaTeX`).
- The actual implementation of LaTeX math that is used in the wiki is jsMath (`http://www.math.union.edu/~dpvc/jsMath`).

5.1 Exercises

Exercise 5.1:1

Write the following formulas in LaTeX.

a) $2 - 3 + 4$

b) $-1 + 0,3$

c) $-5 - (-3) = -5 + 3$

d) $5/2 + 1 > 5/(2 + 1)$

Exercise 5.1:2

Write the following formulas in LaTeX.

a) $3 \cdot 4 \pm 4$

b) $4x^2 - \sqrt{x}$

c) $4 \cdot 3^n \geq n^3$

d) $3 - (5 - 2) = -(-3 + 5 - 2)$

Exercise 5.1:3

Write the following formulas in LaTeX.

a) $\dfrac{x+1}{x^2-1} = \dfrac{1}{x-1}$

b) $\left(\dfrac{5}{x} - 1\right)(1 - x)$

c) $\dfrac{\frac{1}{2}}{\frac{1}{3} + \frac{1}{4}}$

d) $\dfrac{1}{1 + \dfrac{1}{1+x}}$

Exercise 5.1:4

Write the following formulas in LaTeX.

a) $\sin^2 x + \cos x$

b) $\cos v = \cos \dfrac{3\pi}{2}$

c) $\cot 2x = \dfrac{1}{\tan 2x}$

d) $\tan \dfrac{u}{2} = \dfrac{\sin u}{1 + \cos u}$

Exercise 5.1:5

Write the following formulas in LaTeX.

a) $\sqrt{4 + x^2}$

b) $\sqrt[n]{x+y} \neq \sqrt[n]{x} + \sqrt[n]{y}$

c) $\sqrt{\sqrt{3}} = \sqrt[4]{3}$

d) $\left(\sqrt[4]{3}\right)^3 \sqrt[3]{2 + \sqrt{2}}$

Exercise 5.1:6

Write the following formulas in LaTeX.

a) $\ln(4 \cdot 3) = \ln 4 + \ln 3$

b) $\ln(4 - 3) \neq \ln 4 - \ln 3$

c) $\log_2 4 = \dfrac{\ln 4}{\ln 2}$

d) $2^{\log_2 4} = 4$

Exercise 5.1:7

Correct the following LaTeX maths code.

a) `4^{\frac{3}{4}}(1-(3-4)`

b) `2*sqrt(a+b)`

c) `cotx = \dfrac{1}{2}Sin 20^{o}`

5.2 Mathematical text

Contents:

- General advice
- Mixing formulas and text
- Common errors

Learning outcomes:

After this section you will have learned how to:

- Express mathematics
- Explain mathematics

Advice

Explain your solution

The main advice is:

Explain your solution.

The solution must contain not only a statement of the formulae you used, but also a description of how you reasoned. Use words to do this! Imagine you are explaining the solution for a classmate who has difficulty in keeping up with all the steps. You need not explain every little calculation; nonetheless, do not skip important steps. If you simply follow the above advice, you will have done 80% of what is required to provide an adequate solution.

Write good English

Although this is not an assignment in English and, of course, it is the mathematical content that is the most important, you should nonetheless think about things like typos, grammatical errors, etc. If your solution has too many language errors it can give a very negative impression and affect the credibility of the solution. Impression is important!

Make a clean copy of the solution

After you have solved the problem, you should rewrite the solution. Then you can concentrate on its presentation; this even may lead to improvements in the solution itself. A tip is to ask someone else to read your solution to detect ambiguities. It is better to postpone the presentation phase for a later date, so that when you solve the problem the first time you are able to work freely and not commit yourself too early to a specific solution method.

When you enter the solution, be sure to enter it as text, rather than (say) a screen capture from a word processor. It may be easier to write the solution on your own computer using your favourite program, but you should remember that in the next phase your solution is to be included as part of a group project and thus it is important that the solution can be edited, which a screen capture cannot.

A clear answer

Write a clear answer at the end. This is especially important if the solution is long and the answer is scattered in various parts of the text. However, there are problems where the actual solution constitutes the answer (e.g. "Show that ...") and then of course no separate answer at the end is required.

Simplify the answer as far as possible.

Example 1

a) Do not return the answer $\sqrt{8}$, but give $2\sqrt{2}$ as the answer.

b) Do not give the answer as $\sin^2 x + \cos^2 x + 2\sin 2x$ but as $1 + 2\sin 2x$.

c) Do not give the answer as $x = \begin{cases} \pi/4 + n\pi \\ 3\pi/4 + n\pi \end{cases}$ $(n \text{ integer})$ but as $x = \pi/4 + n\pi/2$ $(n \text{ integer})$.

Try and check sub-steps and answers

It can happen that when you solve some equations so called spurious roots turn up as a consequence of the method of solution that is being used. In these cases, explain why spurious roots may have appeared and test the solutions to see which are real solutions and which are spurious roots.

Lost solutions. E.g. a factor on both sides of the equation is cancelled out and one does not realize that the equation obtained by setting this factor to zero provides additional solutions.

Example 2

If you solve the equation $2x^2 - 5x = 0$ by first moving $5x$ to the right-hand side,

$$2x^2 = 5x,$$

and then cancel x on both sides,

$$2x = 5,$$

you will lose the solution $x = 0$.

If you instead factorize the left-hand side,

$$x(2x - 5) = 0,$$

you will be able to read off both solutions: $x = 0$ and $2x - 5 = 0$ (i.e. $x = \frac{5}{2}$).

Go to Exercise 2.1:3 to practice factorization.

An important part of the solution process is to think of reasonable methods to check the answer. For example, one might substitute the solution of an equation back into that equation, and make sure that it really is a solution, because one may well have calculated wrongly (do not confuse this with the investigation of spurious roots). This may also be done for the sub-steps in a solution.

Another point is to assess if the answer is reasonable. Insert values for some of the parameters and ensure that you get the right answer. Eg what happens if $a = 0$, $a = 1$ or a goes to infinity?

Draw clear figures

A figure may explain introduced symbols or reasoning many times better than text, so please use figures. Bear in mind to draw them clearly and do not overload a figure with too many details. It may be better to have several nearly identical figures where each illustrates one point rather than have a great combination-figure which illustrates everything.

Treat formulas as part of the text

It is important that you write your solution in a way that makes it easy for others to follow. To help you we will present some don’ts and dos below to illustrate some tips and common errors that can occur when you mix formulas and text.

Advice about mixing formulas and text:

- Write the explanatory text on previous line
- Think about the punctuation
- Write displayed equations with indentation (or centered)

Formulas should not be seen as something that is extraneous to the text (or vice versa), but both text and formulas are to be integrated together in a linear flow. Therefore don´t write the text inside brackets behind the formulas. Instead, write the explanatory text on the previous line.

Don't

formula (text text text text text text . . .)
formula (text text text text text text . . .)

Do

Text text text text . . .

formula.

Text text text text . . .

formula.

Formulas can be either written as part of the text or as separate formulas. When formulas are separated from the text they appear on their own line and are either centered or slightly indented.

Do

. . . text text text *formula* text.

Text text text

formula

text text text text text text . . .

(Note how the indentation highlights both the explanatory text and the formula.)

A common mistake is to use a colon in front of all displayed formulas.

> **Don't**
>
> ... which provides that:
>
> *formula*
>
> We start...
>
> (Note also there should be a full stop after the formula above.)

As a formula is to be part of the text, it must be treated as part of the sentence. Think therefore about the punctuation. For example, do not forget the full stop after a formula if it ends a sentence.

> **Do**
>
> ... and it is
>
> *formula*.
>
> The next step is ...
>
> (Note the full stop after the formula above.)

A bad habit is excessive numbering. For example, to put a number in front of each step in a solution (numbering should be used for enumeration). The extra digits do not add anything but rather distract. You seldom need to refer back to the individual steps, and when you need to, you can often write something of the sort "when we squared the equation" etc.

> **Don't**
>
> 3. text text text text text text text text ...
>
> *formula*
>
> 4. text text text text text text text text ...

Sometimes one wants to refer back to a separate formula or equation, and in this case it can be given a number (or star) in brackets in the right or left margin.

Do

…text text text text text text text text

$$formula. \tag{1}$$

Text text (1) text text text text text text

$$formula.$$

Text text text text text text text text…

Common errors

Be careful with arrows and similarities

There is a difference between $\Rightarrow$ (implication arrow), $\Leftrightarrow$ (equivalence arrow) and $=$ (equals sign). For two equations that are known a priori to have the same solutions one uses the equivalence arrow $\Leftrightarrow$ to represent this.

However, if we write "Equation 1 $\Rightarrow$ Equation 2," it means that all solutions that Equation 1 has, Equation 2 also has, (but Equation 2 may have more solutions).

Example 3

a) $x + 5 = 3 \quad \Leftrightarrow \quad x = -2$

b) $x^2 - 4x - 1 = 0 \quad \Leftrightarrow \quad (x-2)^2 - 5 = 0$

c) $\sqrt{x} = x - 2 \quad \Rightarrow \quad x = (x-2)^2$

One often does not bother to write the symbol $\Leftrightarrow$ between the different steps in a solution when they are on different lines (and thus the equivalence is implied). It is also often better to use explanatory text instead of arrows between the different steps in the solution. Do not use the implication arrow as a general continuation symbol (in the sense "The next step is").

The equal sign ($=$) is commonly used in two senses, firstly between things that are identical, eg $(x-2)^2 = x^2 - 4x + 4$ which is true for all x, and secondly in equations in which both sides are equal for some x, such as $(x-2)^2 = 4$, which only is satisfied if $x = 0$ or $x = 4$. You should not mix these two different uses of the same symbol.

Example 4

Don't write
$$x^2 - 2x + 1 = (x - 1)^2 = 4$$
when solving the equation $x^2 - 2x + 1 = 4$, since it can lead to misinterpretations.

Write rather
$$x^2 - 2x + 1 = 4 \quad \Leftrightarrow \quad (x - 1)^2 = 4.$$

(There is also a third use of the equals sign, which occurs when defining an expression or for example an operation.)

Simple arrow ($\rightarrow$) is used in mathematics often to handle different kinds of limits: $a \rightarrow \infty$ means that a increases without limit (goes towards infinity). You will probably not need to use a simple arrow in this course.

Do not be careless with brackets

Since multiplication and division have higher priority than addition and subtraction, one must use brackets when addition and/or subtraction is to be carried out first.

Example 5

a) Do not write $1 + x/\cos x$ when you really mean $(1 + x)/\cos x$.

b) Do not write $1 + (1/\sin x)$ when $1 + 1/\sin x$ will do (even if the first expression is, formally, not wrong).

When dealing with algebraic expressions one usually omits the multiplication sign. For example, one almost never would write $4 \cdot x \cdot y \cdot z$ but rather $4xyz$.

This omission of the multiplication gives precedence over other multiplication and division (but not exponentiation). When one therefore writes $1/2R$ it means $1/(2R)$ and not $(1/2)R$. Since this can be a source of misunderstanding, it is not entirely unusual to print the brackets in both situations.

Arguments to the basic elementary functions are written without parentheses. Therefore, you should not write
$$\cos(x), \quad \sin(x), \quad \tan(x), \quad \cot(x), \quad \lg(x) \quad \text{and} \quad \ln(x)$$
but
$$\cos x, \quad \sin x, \quad \tan x, \quad \cot x, \quad \lg x \quad \text{and} \quad \ln x.$$
In fact you should write $\cos 2x$ and not $\cos(2x)$ (since the argument $2x$ is tightly linked together via a juxtaposition), but brackets are necessary when you write $\sin(x + y)$; $\sin(x/2)$ or $(\sin x)^2$ (which you, alternatively, can write as $\sin^2 x$).

Study advice

Useful web sites

- A video course in mathematical writing by Donald Knuth (http://scpd.stanford.edu/knuth/index.jsp). (A compendium accompaning the course is avalable in source form (http://www-cs-faculty.stanford.edu/~knuth/papers/mathwriting.tex.gz) or in excerpts from Google books (http://books.google.com/books?id=dDOehHMbUMcC&printsec=frontcover&dq=inauthor:Donald+inauthor:Ervin+inauthor:Knuth&lr=&ei=JbN1SZfvFZysMqPPhM8M&hl=sv#PPP9,M1).

5.2 Exercises

Exercise 5.2:1

Which of the arrows $\Rightarrow$, $\Leftarrow$ or $\Leftrightarrow$ should you insert between the following equations? (Instead of the question mark.)

a) $\tan x(\sin x + 1) = \tan x \quad ? \quad \sin x + 1 = 1$

b) $\sqrt{x-1} = x + 1 \quad ? \quad x - 1 = (x+1)^2$

c) $x^2 - 6x + 1 = 0 \quad ? \quad (x-3)^2 - 9 + 1 = 0$

Exercise 5.2:2

Criticize the following excerpt from a solution written by a student.

To start with we show that Monikas solution is correct up to the last step where she makes a mistake.

$$cosxtan\frac{3x}{2} = \frac{1}{tanx + cot2x}$$

LHS's denominator is multiplied to both sides.

$$cosxtan\frac{3x}{2}(tanx + cot2x) = 1$$

Exercise 5.2:3

Criticize the following excerpt from a solution written by a student.

$$\frac{1}{2} \times (\sin 4x + \sin 2x) \times \cos x^2 + \frac{1}{2} \times (\sin 4x + \sin 2x)\sin x^2 - \sin 2x \times \sin x^2 = \frac{3}{4}$$

---Now we can factorize the parentheses since they are equal to $(\frac{1}{2}\sin 4x + \frac{1}{2}\sin 2x) \times (\cos x^2 + \sin x^2)$ and the Pythagorean identity $(\cos x^2 + \sin x^2) = 1$ gives that:

$$\frac{1}{2} \times (\sin 4x + \sin 2x) - \sin 2x \times \sin x^2 = \frac{3}{4}$$

Exercise 5.2:4
Criticize the following excerpt from a solution written by a student.

I was in a bit of a hurry when I solved the problem last week... here you have the updated version:

$3 \times 2^x = 4 \times \sqrt[x]{9}$

$3 \times 2^x = 2^2 \times \sqrt[x]{2^2}$

$3 \times 2^x = 2^2 \times 3^{\frac{2}{x}}, x \neq 0$ ($x = 0$ is not a solution)

factor out x from the exponent in the LHS and 2 from RHS

$(3^{\frac{1}{x}} \times 2)^x = (2 \times 3^{\frac{1}{x}})^2$

Exercise 5.2:5
Criticize the following excerpt from a solution written by a student.

Multiply both sides by 1-3tan²x:

$tan(x)(3tan(x) - tan^3(x) + 2sin^2(x) - 2sin^2(x)3tan^2(x) = 0 <=> \frac{sin(x)}{cos(x)}(\frac{3sin(x)}{cos(x)}$

Cancel the factor $sin^2(x)$

$\frac{3}{cos^2(x)} - \frac{sin^2(x)}{cos^2(x)} + 2 - 6\frac{sin^2(x)}{cos^2(x)}$

Exercise 5.2:6
Criticize the following excerpt from a solution written by a student.

A new revision it is. I assume I should update the window with the solution of the group.

My original solution is:

The problem is

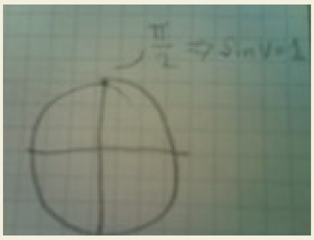
$Sinx + Cosx = \sqrt{2} * Sin2x$

Trigonometry has almost as many formulas as there are stars in the sky. We therefore start to plot a few lines to get a better understanding of what we are looking for.

Exercise 5.2:7

Criticize the following excerpt from a solution written by a student.

We will now solve $\sin 2x$ for x and we replace $2x$ with V. On the unit circle we see that when $V = \frac{\pi}{2}$ we have $\sin V = 1$



Here we must remember to add as many revolutions as possible to $\frac{\pi}{2}$ and get the same answer!

Therefore: $V = \frac{\pi}{2} + 2\pi * n$

Exercise 5.2:8

Criticize the following excerpt from a solution written by a student.

1. $\cos^2 x + \cos^2 2x + \cos^2 3x = \frac{3}{2}$

With the double-angle formula $\cos 2x = 2\cos^2 x - 1$ **we get**

$\cos^2 x = \frac{\cos 2x + 1}{2}$ and $\cos^2 3x = \frac{\cos 6x + 1}{2}$

Inserted into 1 gives:

2. $\frac{\cos 2x + 1}{2} + \cos^2 2x + \frac{\cos 6x + 1}{2} = \frac{3}{2}$

Exercise 5.2:9

Criticize the following excerpt from a solution written by a student.

$\cos x \tan(3x/2) = 1/(\tan x + \cot 2x)$

I replace tan and cot with their corresponding cossine and sine.

$\cos x * (\sin(3x/2)/\cos(3x/2)) = 1/((\sin x/\cos x) + (\cos 2x/\sin 2x) - - >$

$\cos x * (\sin(3x/2)/\cos(3x/2)) * ((\sin x/\cos x) + (\cos 2x/\sin 2x)) = 1$

Answers to the exercises

Numerical calculations

1.1:1 a) -7 b) 1 c) 11 d) 1

1.1:2 a) 0 b) -1 c) -25 d) -19

1.1:3 a) natural numbers, integers, rational numbers
b) integers, rational numbers
c) natural numbers, integers, rational numbers
d) integers, rational numbers
e) integers, rational numbers
f) natural numbers, integers, rational numbers

1.1:4 a) rational numbers
b) natural numbers, integers, rational numbers
c) irrational numbers
d) natural numbers, integers, rational numbers
e) irrational numbers
f) irrational numbers

1.1:5 a) $\frac{3}{5} < \frac{5}{3} < 2 < \frac{7}{3}$
b) $-\frac{1}{2} < -\frac{1}{3} < -\frac{3}{10} < -\frac{1}{5}$
c) $\frac{1}{2} < \frac{3}{5} < \frac{21}{34} < \frac{5}{8} < \frac{2}{3}$

1.1:6 a) 1.167 b) 2.250 c) 0.286
d) 1.414

1.1:7 a) rational, $\frac{314}{100} = \frac{157}{50}$
b) rational, $\frac{31413}{9999} = \frac{10471}{3333}$
c) rational, $\frac{1999}{9990}$
d) irrational

1.2:1 a) $\frac{93}{28}$ b) $\frac{3}{35}$ c) $-\frac{7}{30}$ d) $\frac{47}{60}$
e) $\frac{47}{84}$

1.2:2 a) 30 b) 8 c) 84 d) 225

1.2:3 a) $\frac{19}{100}$ b) $\frac{1}{240}$

1.2:4 a) $\frac{6}{7}$ b) $\frac{16}{21}$ c) $\frac{1}{6}$

1.2:5 a) $\frac{105}{4}$ b) -5 c) $\frac{8}{55}$

1.2:6 $\frac{152}{35}$

1.3:1 a) 72 b) 3 c) -125 d) $\frac{27}{8}$

1.3:2 a) 2^6 b) 2^{-2} c) 2^0

1.3:3 a) 3^{-1} b) 3^5 c) 3^4 d) 3^{-3}
e) 3^{-3}

1.3:4 a) 4 b) 3 c) 625 d) 16 e) $\frac{1}{3750}$

1.3:5 a) 2 b) $\frac{1}{2}$ c) 27 d) 2209 e) 9
f) $\frac{25}{3}$

1.3:6 a) $256^{1/3} > 200^{1/3}$
b) $0{,}4^{-3} > 0{,}5^{-3}$
c) $0{,}2^5 > 0{,}2^7$
d) $\left(5^{1/3}\right)^4 > 400^{1/3}$
e) $125^{1/2} > 625^{1/3}$
f) $3^{40} > 2^{56}$

Algebra

2.1:1 a) $3x^2 - 3x$
b) $xy + x^2y - x^3y$
c) $-4x^2 + x^2y^2$
d) $x^3y - x^2y + x^3y^2$
e) $x^2 - 14x + 49$
f) $16y^2 + 40y + 25$
g) $9x^6 - 6x^3y^2 + y^4$
h) $9x^{10} + 30x^8 + 25x^6$

2.1:2 a) $-5x^2 + 20$ b) $10x - 11$ c) $54x$
d) $81x^8 - 16$ e) $2a^2 + 2b^2$

2.1:3 a) $(x+6)(x-6)$
b) $5(x+2)(x-2)$
c) $(x+3)^2$
d) $(x-5)^2$

e) $-2x(x+3)(x-3)$
f) $(4x+1)^2$

2.1:4 a) 5 in front of x^2, 3 in front of x
b) 2 in front of x^2, 1 in front of x
c) 6 in front of x^2, 2 in front of x

2.1:5 a) $\dfrac{1}{1-x}$ b) $-\dfrac{1}{y(y+2)}$
c) $3(x-2)(x-1)$ d) $\dfrac{2(y+2)}{y^2+4}$

2.1:6 a) $2y$ b) $\dfrac{-x+12}{(x-2)(x+3)}$
c) $\dfrac{b}{a(a-b)}$ d) $\dfrac{a(a+b)}{4b}$

2.1:7 a) $\dfrac{4}{(x+3)(x+5)}$
b) $\dfrac{x^4-x^3+x^2+x-1}{x^2(x-1)}$
c) $\dfrac{ax(a+1-x)}{(a+1)^2}$

2.1:8 a) $\dfrac{x}{(x+3)(x+1)}$ b) $\dfrac{2(x-3)}{x}$
c) $\dfrac{x+2}{2x+3}$

2.2:1 a) $x=1$ b) $x=6$ c) $x=-\frac{3}{2}$
d) $x=-\frac{13}{3}$

2.2:2 a) $x=1$ b) $x=\frac{5}{3}$ c) $x=2$
d) $x=-2$

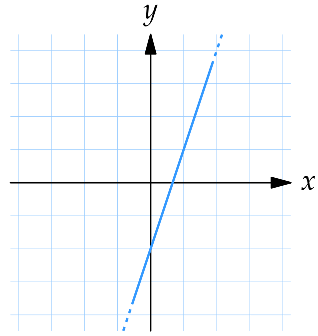
2.2:3 a) $x=9$ b) $x=\frac{7}{5}$ c) $x=\frac{4}{5}$
d) $x=\frac{1}{2}$

2.2:4 a) $-2x+y=3$ b) $y=-\frac{3}{4}x+\frac{5}{4}$

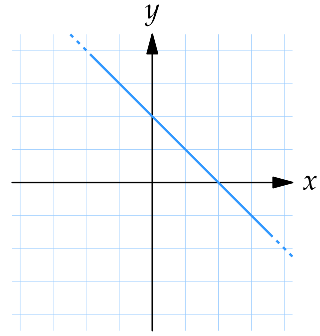
2.2:5 a) $y=-3x+9$ b) $y=-3x+1$
c) $y=3x+5$ d) $y=-\frac{1}{2}x+5$
e) $k=\frac{8}{5}$

2.2:6 a) $(-\frac{5}{3},0)$ b) $(0,5)$ c) $(0,-\frac{6}{5})$
d) $(12,-13)$ e) $(-\frac{1}{4},\frac{3}{2})$

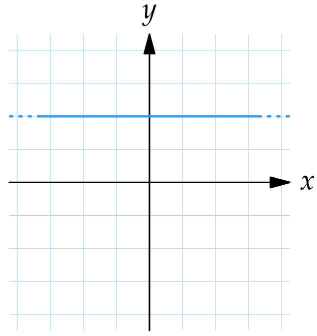
2.2:7 a)



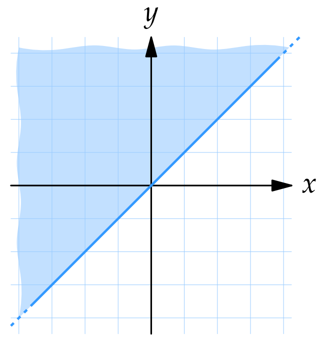
b)



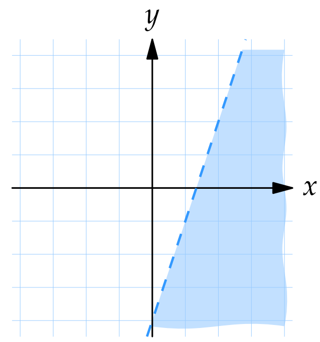
c)

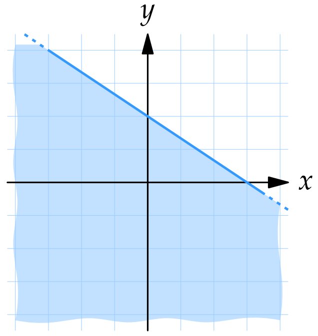


2.2:8 a)



b)



c) 

2.2:9 a) 4 units b) 5 units
c) 6 units

2.3:1 a) $(x-1)^2-1$ b) $(x+1)^2-2$
c) $-(x-1)^2+6$ d) $\left(x+\frac{5}{2}\right)^2-\frac{13}{4}$

2.3:2 a) $\begin{cases} x_1 = 1 \\ x_2 = 3 \end{cases}$ b) $\begin{cases} y_1 = -5 \\ y_2 = 3 \end{cases}$

c) no (real) solution

d) $\begin{cases} x_1 = \frac{1}{2} \\ x_2 = \frac{13}{2} \end{cases}$ e) $\begin{cases} x_1 = -1 \\ x_2 = \frac{3}{5} \end{cases}$

f) $\begin{cases} x_1 = \frac{4}{3} \\ x_2 = 2 \end{cases}$

2.3:3 a) $\begin{cases} x_1 = 0 \\ x_2 = -3 \end{cases}$ b) $\begin{cases} x_1 = 3 \\ x_2 = -5 \end{cases}$

c) $\begin{cases} x_1 = \frac{2}{3} \\ x_2 = -8 \end{cases}$ d) $\begin{cases} x_1 = 0 \\ x_2 = 12 \end{cases}$

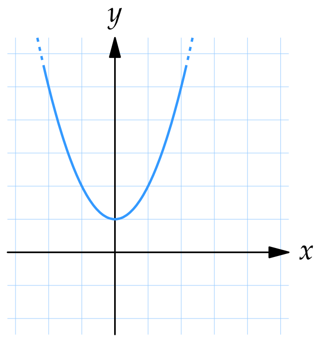
e) $\begin{cases} x_1 = -3 \\ x_2 = 8 \end{cases}$ f) $\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 2 \end{cases}$

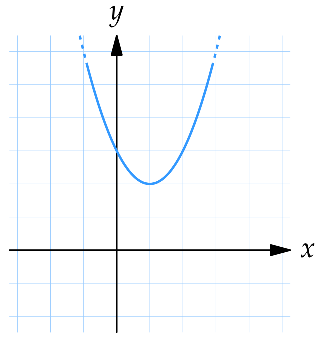
2.3:4 a) $ax^2-ax-2a=0\,,$
b) $ax^2-2ax-2a=0\,,$
c) $ax^2-(3+\sqrt{3}\,)ax+3\sqrt{3}\,a=0\,,$
where $a \neq 0$ is a constant.

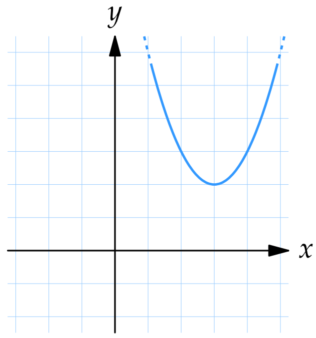
2.3:5 a) For example $x^2+14x+49=0$
b) $3<x<4$
c) $b=-5$

2.3:6 a) 0 b) -2 c) $\frac{3}{4}$

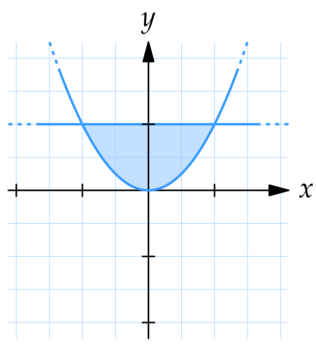
2.3:7 a) 1 b) $-\frac{7}{4}$ c) no max

2.3:8 a) 

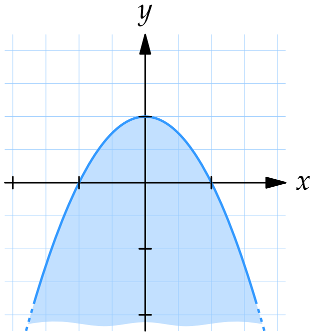
b) 

c) 

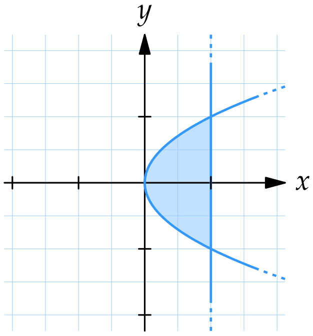
2.3:9 a) $(-1,0)$ and $(1,0)$
b) $(2,0)$ and $(3,0)$
c) $(1,0)$ and $(3,0)$

2.3:10 a) 

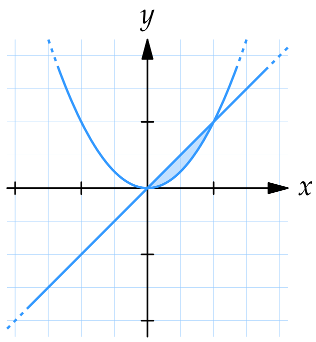
b)



c)



d)



Roots and logarithms

3.1:1 a) $2^{1/2}$ b) $7^{5/2}$ c) $3^{4/3}$ d) $3^{1/4}$

3.1:2 a) 3 b) 3 c) not defined
d) $5^{11/6}$ e) 12 f) 2 g) -5

3.1:3 a) 3 b) $4\sqrt{3}/3$ c) $2\sqrt{5}$
d) $2-\sqrt{2}$

3.1:4 a) 0,4 b) 0,3 c) $-4\sqrt{2}$ d) $2\sqrt{3}$

3.1:5 a) $\sqrt{3}/3$ b) $7^{2/3}/7$ c) $3-\sqrt{7}$
d) $(\sqrt{17}+\sqrt{13})/4$

3.1:6 a) $6+2\sqrt{2}+3\sqrt{5}+\sqrt{10}$
b) $-(5+4\sqrt{3})/23$
c) $\frac{2}{3}\sqrt{6}+\frac{2}{3}\sqrt{3}-\frac{2}{5}\sqrt{10}-\frac{2}{5}\sqrt{5}$
d) $(5\sqrt{3}+7\sqrt{2}-\sqrt{6}-12)/23$

3.1:7 a) $\sqrt{5}-\sqrt{7}$ b) $-\sqrt{35}$ c) $\sqrt{17}$

3.1:8 a) $\sqrt[3]{6} > \sqrt[3]{5}$
b) $7 > \sqrt{7}$
c) $\sqrt{7} > 2{,}5$
d) $\sqrt[3]{2}\cdot 3 > \sqrt{2}\left(\sqrt[4]{3}\right)^3$

3.2:1 $x = 5$

3.2:2 $x = 1$

3.2:3 $\begin{cases} x_1 = 3 \\ x_2 = 4 \end{cases}$

3.2:4 no solution

3.2:5 $x = 1$

3.2:6 $x = \frac{5}{4}$

3.3:1 a) $x = 3$ b) $x = -1$ c) $x = -2$
d) $x = 4$

3.3:2 a) -1 b) 4 c) -3 d) 0 e) 2
f) 3 g) 10 h) -2

3.3:3 a) 3 b) $-\frac{1}{2}$ c) -3 d) $\frac{7}{3}$ e) 4
f) -2 g) 1 h) $\frac{5}{2}$

3.3:4 a) 1 b) 0 c) $-\frac{1}{2}\lg 3$

3.3:5 a) 5 b) 0 c) 0 d) 0 e) -2
f) e^2

3.3:6 a) 1.262 b) 1.663 c) 4.762

3.4:1 a) $x = \ln 13$ b) $x = \dfrac{\ln 2 - \ln 13}{1+\ln 3}$
c) $x = \dfrac{\ln 7 - \ln 3}{1-\ln 2}$

3.4:2 a) $\begin{cases} x_1 = \sqrt{2} \\ x_2 = -\sqrt{2} \end{cases}$
b) $x = \ln \dfrac{\sqrt{17}-1}{2}$
c) no solution

3.4:3 a) $x = -\dfrac{1}{\ln 2} \pm \sqrt{\left(\dfrac{1}{\ln 2}\right)^2 - 1}$

b) $x = \frac{5}{2}$

c) $x = 1$

Trigonometry

4.1:1 a) 90° and $\pi/2$ rad

b) 135° and $3\pi/4$ rad

c) -240° and $-4\pi/3$ rad

d) 2910° and $97\pi/6$ rad

4.1:2 a) $\pi/4$ rad

b) $3\pi/4$ rad

c) $-7\pi/20$ rad

d) $3\pi/2$ rad

4.1:3 a) $x = 50$ b) $x = 5$ c) $x = 15$

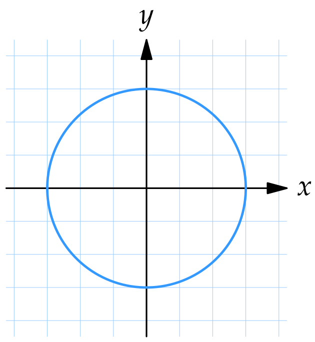
4.1:4 a) 5 units b) $\sqrt{61}$ units

c) $(2, 0)$

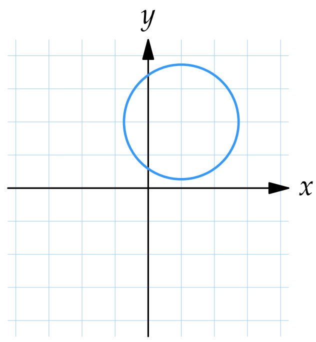
4.1:5 a) $(x-1)^2 + (y-2)^2 = 4$

b) $(x-2)^2 + (y+1)^2 = 13$

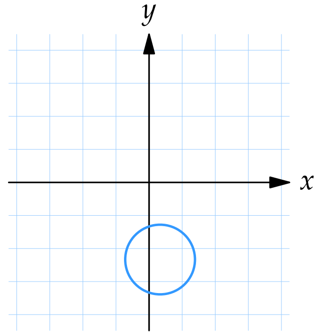
4.1:6 a)



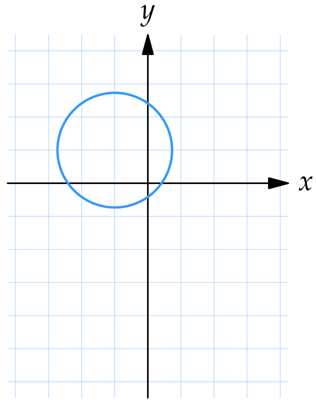
b)



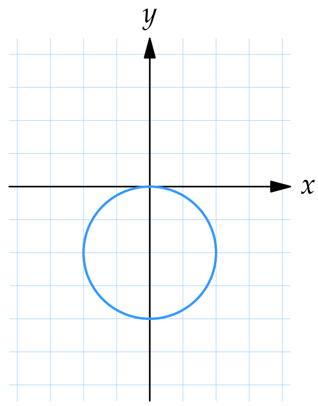
c)



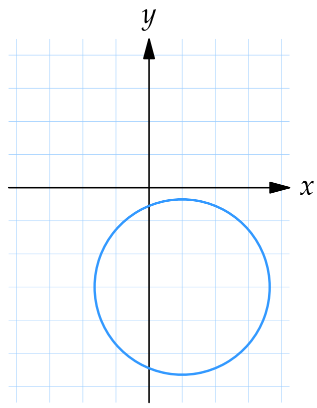
4.1:7 a)

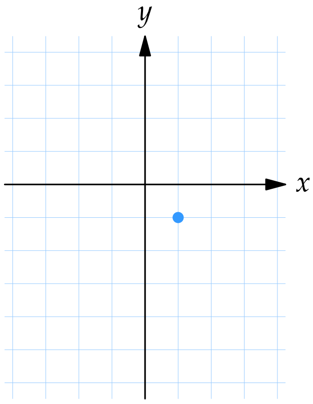


b)



c)



d) 

4.1:8 $10/\pi$ revolutions ≈ 3.2 revolutions

4.1:9 $32\pi/3\ \text{cm}^2 \approx 33.5\ \text{cm}^2$

4.1:10 $x = 9$ dm

4.2:1 a) $x = 13 \cdot \tan 27^\circ \approx 6.62$
b) $x = 25 \cdot \cos 32^\circ \approx 21.2$
c) $x = 14/\tan 40^\circ \approx 16.7$
d) $x = 16/\cos 20^\circ \approx 17.0$
e) $x = 11/\sin 35^\circ \approx 19.2$
f) $x = 19/\tan 50^\circ \approx 15.9$

4.2:2 a) $\tan v = \frac{2}{5}$ b) $\sin v = \frac{7}{11}$
c) $\cos v = \frac{5}{7}$ d) $\sin v = \frac{3}{5}$
e) $v = 30^\circ$ f) $\sin(v/2) = \frac{1}{3}$

4.2:3 a) -1 b) 1 c) 0 d) 0
e) $1/\sqrt{2}$ f) $\sqrt{3}/2$

4.2:4 a) $\sqrt{3}/2$ b) $\frac{1}{2}$ c) -1 d) 0
e) $1/\sqrt{3}$ f) $\sqrt{3}$

4.2:5 a) $-1/\sqrt{2}$ b) 1 c) $\sqrt{3}/2$
d) -1

4.2:6 $x = \sqrt{3} - 1$

4.2:7 Width of the river $= \dfrac{10}{\sqrt{3}-1}$ m $\approx$ 13.7 m.

4.2:8 $\ell \cos\gamma = a\cos\alpha - b\cos\beta$

4.2:9 Distance $= \sqrt{205 - 48\sqrt{3}} \approx 11.0$ km

4.3:1 a) $v = 9\pi/5$ b) $v = 6\pi/7$
c) $v = 9\pi/7$

4.3:2 a) $v = \pi/2$ b) $v = 3\pi/5$

4.3:3 a) $-a$ b) a c) $\sqrt{1-a^2}$
d) $\sqrt{1-a^2}$ e) $-a$
f) $\dfrac{\sqrt{3}}{2}\sqrt{1-a^2} + \dfrac{1}{2} \cdot a$

4.3:4 a) $1 - b^2$ b) $\sqrt{1-b^2}$
c) $2b\sqrt{1-b^2}$ d) $2b^2 - 1$
e) $\sqrt{1-b^2} \cdot \dfrac{1}{\sqrt{2}} + b \cdot \dfrac{1}{\sqrt{2}}$
f) $b \cdot \dfrac{1}{2} + \sqrt{1-b^2} \cdot \dfrac{\sqrt{3}}{2}$

4.3:5 $\cos v = \dfrac{2\sqrt{6}}{7}$, $\tan v = \dfrac{5}{2\sqrt{6}}$

4.3:6 a) $\sin v = -\dfrac{\sqrt{7}}{4}$, $\tan v = -\dfrac{\sqrt{7}}{3}$
b) $\cos v = -\dfrac{\sqrt{91}}{10}$, $\tan v = -\dfrac{3}{\sqrt{91}}$
c) $\sin v = -\dfrac{3}{\sqrt{10}}$, $\cos v = -\dfrac{1}{\sqrt{10}}$

4.3:7 a) $\sin(x+y) = \dfrac{4\sqrt{2}+\sqrt{5}}{9}$
b) $\sin(x+y) = \dfrac{3\sqrt{21}+8}{25}$

4.4:1 a) $v = \pi/6$, $v = 5\pi/6$
b) $v = \pi/3$, $v = 5\pi/3$
c) $v = \pi/2$
d) $v = \pi/4$, $v = 5\pi/4$
e) no solution
f) $v = 11\pi/6$, $v = 7\pi/6$
g) $v = 5\pi/6$, $v = 11\pi/6$

4.4:2 a) $\begin{cases} x = \pi/3 + 2n\pi \\ x = 2\pi/3 + 2n\pi \end{cases}$
b) $\begin{cases} x = \pi/3 + 2n\pi \\ x = 5\pi/3 + 2n\pi \end{cases}$

c) $x = n\pi$

d) $\begin{cases} x = \pi/20 + 2n\pi/5 \\ x = 3\pi/20 + 2n\pi/5 \end{cases}$

e) $\begin{cases} x = \pi/30 + 2n\pi/5 \\ x = \pi/6 + 2n\pi/5 \end{cases}$

f) $\begin{cases} x = \pi/4 + 2n\pi/3 \\ x = 5\pi/12 + 2n\pi/3 \end{cases}$

4.4:3 a) $\begin{cases} x = \pi/6 + 2n\pi \\ x = 11\pi/6 + 2n\pi \end{cases}$

b) $\begin{cases} x = \pi/5 + 2n\pi \\ x = 4\pi/5 + 2n\pi \end{cases}$

c) $\begin{cases} x = 25^\circ + n \cdot 360^\circ \\ x = 75^\circ + n \cdot 360^\circ \end{cases}$

d) $\begin{cases} x = 5^\circ + n \cdot 120^\circ \\ x = 55^\circ + n \cdot 120^\circ \end{cases}$

4.4:4 $v_1 = 50^\circ,\ \ v_2 = 120^\circ,\ \ v_3 = 230^\circ,$
$v_4 = 300^\circ$

4.4:5 a) $\begin{cases} x = n\pi \\ x = \pi/4 + n\pi/2 \end{cases}$

b) $x = n\pi/3$

c) $\begin{cases} x = \pi/20 + n\pi/2 \\ x = -\pi/30 + n\pi/3 \end{cases}$

4.4:6 a) $x = n\pi$

b) $\begin{cases} x = \pi/4 + 2n\pi \\ x = \pi/2 + n\pi \\ x = 3\pi/4 + 2n\pi \end{cases}$

c) $\begin{cases} x = 2n\pi/3 \\ x = \pi + 2n\pi \end{cases}$

4.4:7 a) $\begin{cases} x = \pi/6 + 2n\pi \\ x = 5\pi/6 + 2n\pi \\ x = 3\pi/2 + 2n\pi \end{cases}$

b) $x = \pm\pi/3 + 2n\pi$

c) $\begin{cases} x = \pi/2 + 2n\pi \\ x = \pi/14 + 2n\pi/7 \end{cases}$

4.4:8 a) $\begin{cases} x = \pi/4 + 2n\pi \\ x = \pi/2 + n\pi \\ x = 3\pi/4 + 2n\pi \end{cases}$

b) $x = \pi/3 + n\pi$

c) $\begin{cases} x = n\pi \\ x = 3\pi/4 + n\pi \end{cases}$

Written mathematics

5.1:1 a) `2-3+4`
b) `-1+0,3`
c) `-5-(-3)=-5+3`
d) `5/2+1 > 5/(2+1)`

5.1:2 a) `3\cdot 4\pm 4`
b) `4x^2-\sqrt{x}`
c) `4\cdot 3^n\ge n^3`
d) `3-(5-2)=-(-3+5-2)`

5.1:3 a) `\dfrac{x+1}{x^2-1}=\dfrac{1}{x-1}`
b) `\left(\dfrac{5}{x}-1\right)(1-x)`
c) `\dfrac{\frac{1}{2}}{\frac{1}{3}`
`+\frac{1}{4}}`
d) `\dfrac{1}{1+\dfrac{1}{1+x}}`

5.1:4 a) `\sin^2 x+\cos x`
b) `\cos v=\cos\dfrac{3\pi}{2}`
c) `\cot 2x=\dfrac{1}{\tan 2x}`
d) `\tan\dfrac{u}{2}=\dfrac{\sin u}`
`{1+\cos u}`

5.1:5 a) `\sqrt{4+x^2}`
b) `\sqrt[n]{x+y}\ne\sqrt[n]{x}`
`+\sqrt[n]{y}`
c) `\sqrt{\sqrt{3}} = \sqrt[4]{3}`
d) `\left(\sqrt[4]{3}\right)^3`
`\sqrt[3]{2+\sqrt{2}}`

5.1:6 a) `\ln(4\cdot 3)=\ln 4+\ln 3`
b) `\ln(4-3)\ne \ln 4-\ln 3`
c) `\log_{2}4=\dfrac{\ln 4}{\ln 2}`
d) `2^{\log_{2}4}=4`

5.1:7 a) `4^{3/4}(1-(3-4))`
b) `2\sqrt{a+b}`
c) `\cot x=\frac{1}{2}\sin 20^{\circ}`

5.2:1 a) $\Leftarrow$ b) $\Rightarrow$ c) $\Leftrightarrow$