Rigid Body Dynamics (SG2150)
Exam, 2017-10-27, 14.00-18.00

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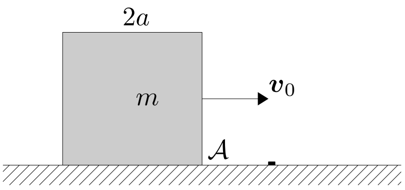
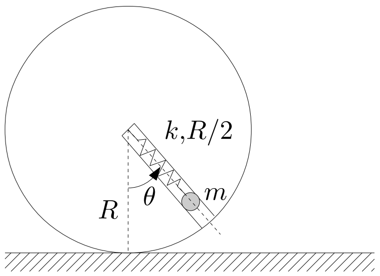
Each problem gives a maximum of 3 points, so that the total maximum is 18. Grading: 1–3 F; 4–5 FX; 6: E; 7–9 D; 10–12 C; 13–15 B; 16–18 A.

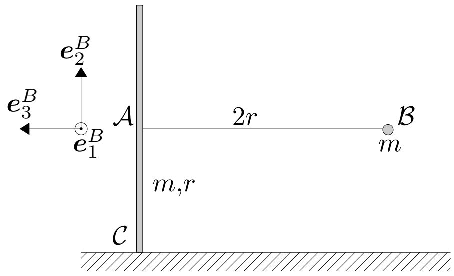
Allowed equipment: Handbook of mathematics and physics. One one-sided A4 page with your own compilation of formulae.



Problem 1. A homogeneous cube of mass m and with side length $2a$ is sliding on a smooth horizontal plane with initial velocity v_0 to the right, when the lower right corner $\mathcal{A}$ encounters a low stop in the plane. The impact causes the corner $\mathcal{A}$ to loose all velocity. Compute the angular velocity of the cube immediately after the impact.
If the initial velocity $v_0 \leq v_c = \sqrt{8ga/3}$, then the corner $\mathcal{A}$ will remain in contact with the stop after impact for tilt angles up to $\pi/4$. Show that $v_0 = v_c$ is enough initial velocity to reach tilt angle $\pi/4$ (which will make the cube overturn instead of falling back).



Problem 2. A light cylindrical shell of radius R is rolling on a rough horizontal plane. On the inside surface of the shell a thin hollow radial light tube of length R is attached. Inside the smooth tube, a particle of mass m can slide, and the particle is connected with the inner end point of the tube through a spring with spring constant $k = 4mg/R$ and unstressed length $R/2$.
Find a stable equilibrium solution for the system, and find the frequencies of small oscillations about this equilibrium.



Problem 3. A thin homogeneous circular disc of mass m and radius r is rolling on a horizontal rough plane. At the centre $\mathcal{A}$ of the disc, a light perpendicular axis $\mathcal{AB}$ of length $2r$ is attached, and at the end point $\mathcal{B}$ a point mass m is attached. We introduce a basis triad B such that $\boldsymbol{e}_3^B$ points in the direction $\mathcal{BA}$, $\boldsymbol{e}_1^B$ is horizontal, and $\boldsymbol{e}_2^B$ has a positive vertical up component. Let the angular velocity of the body be $\boldsymbol{\omega} = \omega_1 \boldsymbol{e}_1^B + \omega_2 \boldsymbol{e}_2^B + \omega_3 \boldsymbol{e}_3^B$.
Now assume that the motion happens to be such that the axis $\mathcal{AB}$ remains horizontal. Using this assumption show that

- $\omega_1 = 0$ for all time.
- Both ω_2 and ω_3 are constant in time.
- There is a relation giving $\omega_3(\omega_2)$.

If possible, also try to describe the path the contact point $\mathcal{C}$ traces out on the plane.

Problem 4. For the rigid body (disc + point mass) of Problem 3, compute all three principal moments of inertia about the contact point $\mathcal{C}$.

Problem 5. *Spherical coordinates.* Starting with the fixed triad $O : [\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z] = [\boldsymbol{e}_1^O, \boldsymbol{e}_2^O, \boldsymbol{e}_3^O]$, first perform a simple rotation about the $\boldsymbol{e}_3^O$ axis by an angle φ, followed by a simple rotation about the new 2-axis an angle θ to get a final triad A. Relabel the basis vectors in the triad A as $\boldsymbol{e}_r = \boldsymbol{e}_3^A$, $\boldsymbol{e}_\theta = \boldsymbol{e}_1^A$, and $\boldsymbol{e}_\varphi = \boldsymbol{e}_2^A$, and finally let the position vector of a point be $\boldsymbol{r} = r\boldsymbol{e}_r(\theta, \varphi)$. The variables $r(t), \theta(t), \varphi(t)$ are the spherical coordinates of the point. Use this construction the compute the angular velocity ${}^O\boldsymbol{\omega}^A$ of the triad A and then use this to compute the velocity

$$\boldsymbol{v} = \frac{{}^O d\boldsymbol{r}}{dt} = \frac{{}^A d\boldsymbol{r}}{dt} + {}^O\boldsymbol{\omega}^A \times \boldsymbol{r}.$$

of the point. Both vectors ${}^O\boldsymbol{\omega}^A$ and $\boldsymbol{v}$ should be expressed using the spherical basis vectors $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\varphi$.

Problem 6. *Routh's method for cyclic coordinates.* In a system with two generalised coordinates q_1 and q_2, let q_2 be cyclic so the Lagrange function can be written $L(q_1, \dot{q}_1, \dot{q}_2, t)$. The generalised momentum corresponding to q_2 is

$$p_2 = \frac{\partial L}{\partial \dot{q}_2}(q_1, \dot{q}_1, \dot{q}_2, t).$$

Solve this relation for $\dot{q}_2$ to get $\dot{q}_2 = \dot{q}_2(q_1, \dot{q}_1, p_2, t)$. Define the Routh function

$$R(q_1, \dot{q}_1, p_2, t) = L(q_1, \dot{q}_1, \dot{q}_2, t) - p_2 \dot{q}_2.$$

Show that for any values of $q_1, \dot{q}_1, p_2, t$ with corresponding $\dot{q}_2$, we have

$$\frac{\partial R}{\partial \dot{q}_1} = \frac{\partial L}{\partial \dot{q}_1} \text{ and } \frac{\partial R}{\partial q_1} = \frac{\partial L}{\partial q_1}$$

(Remember that when taking partial derivatives, L is a function of $q_1, \dot{q}_1, \dot{q}_2, t$, whereas R is a function of $q_1, \dot{q}_1, p_2, t$!)