Rigid Body Dynamics (SG2150)
Solution to Exam, 2017-10-27, 14.00-18.00

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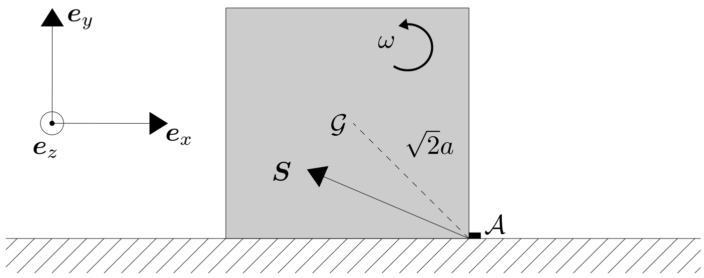
Problem 1.



First study the impact. Since the impact impulse $\boldsymbol{S}$ has no moment of force about the point $\mathcal{A}$, the angular momentum about $\mathcal{A}$ is unchanged. Before impact we have

$$L_{\mathcal{A}_z,i} = -amv_0$$

Immediately after the impact $\mathcal{A}$ is an instantaneous centre of rotation, so

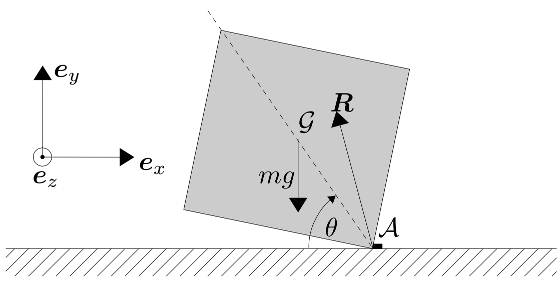
$$L_{\mathcal{A}_z,\mathrm{f}} = J_{\mathcal{A}_z}\omega_{\mathrm{f}}.$$

The moment of inertia of the cube about the corner $\mathcal{A}$ is

$$J_{\mathcal{A}_z} = J_{\mathcal{G}_z} + m(\sqrt{2}a)^2 = \frac{8}{3}ma^2.$$

This gives

$$\boldsymbol{\omega}_{\mathrm{f}} = -\frac{3v_0}{8a}\boldsymbol{e}_z.$$



Next study motion after impact. Assuming the corner $\mathcal{A}$ remains in contact with the stop, which makes it remain an instantaneous centre of rotation, the kinetic energy is

$$T = \frac{8}{3}ma^2\frac{\dot{\theta}^2}{2}.$$

The only force doing work is the conservative gravity force, and the potential energy is

$$V = mg\sqrt{2}a\sin(\theta).$$

Thus

$$T + V = \frac{8}{3}ma^2\frac{\dot{\theta}^2}{2} + mg\sqrt{2}a\sin(\theta) = mga\left[\frac{4}{3}\frac{a\dot{\theta}^2}{g} + \sqrt{2}\sin(\theta)\right] = E \text{ const.}$$

The initial conditions $\dot{\theta} = 3v_0/(8a)$, $\theta = \pi/4$ calibrates the constant to

$$E = mga\left[\frac{3}{16}\frac{v_0^2}{ga} + 1\right].$$

In particular, with $v_0 = v_c$, the energy equation gives

$$\frac{4}{3}\frac{a\dot{\theta}^2}{g} = \frac{3}{16}\frac{8}{3} + 1 - \sqrt{2}\sin(\theta) = \frac{3}{2} - \sqrt{2}\sin(\theta).$$

For $\pi/4 < \theta < \pi/2$, the right hand side remains positive, so $\dot{\theta}$ remains positive, and θ will keep increasing until it reaches $\pi/2$.

Note that we really should check that the contact forces from the plane and the stop are pushing (rather than pulling). For the impact impulses, we will find

$$\frac{S_x}{mv_0} = -\frac{5}{8}, \quad \frac{S_y}{mv_0} = \frac{3}{8}$$

which have the correct signs. For the motion after impact, and assuming contact with both the plane and the stop, we will find

$$\frac{R_x}{mg} = -\sqrt{2}\frac{9}{64}\cos(\theta)\left[8\sqrt{2}\sin(\theta) - \frac{16}{3} - \frac{v_0^2}{ga}\right], \quad \frac{R_y}{mg} = \frac{1}{4} + \sqrt{2}\frac{9}{64}\sin(\theta)\left[8\sqrt{2}\sin(\theta) - \frac{16}{3} - \frac{v_0^2}{ga}\right].$$

Since we want $R_x < 0$ in the interval $\pi/4 < \theta < \pi/2$, we must have

$$\frac{v_0^2}{ga} \leq \frac{8}{3} = \frac{v_c^2}{ga}.$$

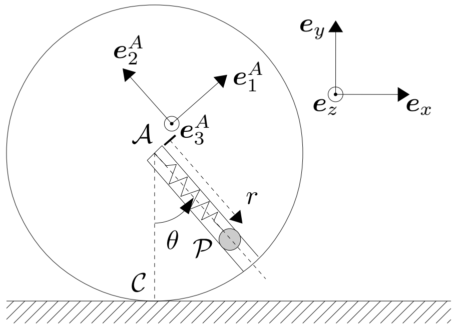
We then easily see that $R_y > 0$ as well. Upon passing $\theta = \pi/2$, the corner $\mathcal{A}$ will start sliding left.

Thus desired motion of overturning with the corner in contact with the stop is only possible for the rather narrow interval

$$\frac{16}{3}\left(\sqrt{2} - 1\right) < \frac{v_0^2}{ga} \leq \frac{8}{3},$$

where the left inequality comes from having enough energy, and the right from the contact force requirement.

Problem 2.



Use the coordinate r of the particle along the tube, and the angle θ as generalised coordinates.

To find equilibrium points, we study the potential energy. The gravity force and the spring force contributes. We find

$$V = -mgr\cos(\theta) + \frac{k}{2}\left(r - \frac{R}{2}\right)^2 = mg\left[-r\cos(\theta) + \frac{2}{R}\left(r - \frac{R}{2}\right)^2\right].$$

using $k = 4mg/R$. Equilibrium points are found from

$$\begin{aligned}\frac{\partial V}{\partial r} &= mg\left[-\cos(\theta) + \frac{4}{R}\left(r - \frac{R}{2}\right)\right] = 0\\ \frac{\partial V}{\partial \theta} &= mgr\sin(\theta) = 0.\end{aligned}$$

The second equation gives either $r = 0$ (where the first equation becomes $-\cos(\theta) - 2 = 0$ with no solutions) or $\sin(\theta) = 0$. We find

$$\theta_1 = 0,\ r_1 = \frac{3R}{4},\ \text{or}\ \theta_2 = \pi,\ r_2 = \frac{R}{4}$$

as the two equilibrium points.

At the second equilibrium point

$$\frac{\partial^2 V}{\partial \theta^2} = mgr_2\cos(\theta_2) = -\frac{mgR}{4} < 0,$$

so the stiffness matrix $\boldsymbol{K}_2$ is not positive definite, and equilibrium point 2 is not stable.

At the first equilibrium point, the stiffness matrix is

$$\boldsymbol{K}_1 = \frac{mg}{R}\begin{bmatrix}4 & 0\\ 0 & \frac{3}{4}R^2\end{bmatrix}.$$

To compute the kinetic energy, we need the velocity of the particle. Both the cylinder and the triad A has angular velocity $\boldsymbol{\omega} = \dot{\theta}\boldsymbol{e}_z = \dot{\theta}\boldsymbol{e}_3^A$. Using rolling, so the contact point is an instantaneous centre of rotation for the cylinder, we find for the cylinder centre point $\mathcal{A}$

$$\boldsymbol{v}_{\mathcal{A}} = \boldsymbol{\omega} \times \boldsymbol{r}_{\mathcal{CA}} = -R\dot{\theta}\boldsymbol{e}_x.$$

Using $\boldsymbol{r}_{\mathcal{AP}} = -r\boldsymbol{e}_2^A$, we also have

$$\boldsymbol{v}_{\mathcal{P}} = \boldsymbol{v}_{\mathcal{A}} + \dot{\boldsymbol{r}}_{\mathcal{AP}} = \boldsymbol{v}_{\mathcal{A}} + \frac{{}^A d\boldsymbol{r}_{\mathcal{AP}}}{dt} + \boldsymbol{\omega} \times \boldsymbol{r}_{\mathcal{AP}} = -R\dot{\theta}\boldsymbol{e}_x - \dot{r}\boldsymbol{e}_2^A + r\dot{\theta}\boldsymbol{e}_1^A.$$

Since we are considering small oscillations about the equilibrium point (r_1, θ_1) it is enough to compute the velocity in the equilibrium point configuration, where $r = 3R/4$, $\boldsymbol{e}_1^A = \boldsymbol{e}_x$, and $\boldsymbol{e}_2^A = \boldsymbol{e}_y$, so

$$\boldsymbol{v}_{\mathcal{P}} = \left(-R + \frac{3}{4}R\right)\dot{\theta}\boldsymbol{e}_x - \dot{r}\boldsymbol{e}_y = -\frac{R}{4}\dot{\theta}\boldsymbol{e}_x - \dot{r}\boldsymbol{e}_y.$$

The kinetic energy at the equilibrium point is

$$T_1 = \frac{m}{2}|\boldsymbol{v}_{\mathcal{P}}|^2 = \frac{m}{2}\left(\frac{R^2}{16}\dot{\theta}^2 + \dot{r}^2\right).$$

This gives the mass matrix

$$\boldsymbol{M}_1 = m\begin{bmatrix}1 & 0\\ 0 & \frac{1}{16}R^2\end{bmatrix}.$$

The eigenvalue problem now becomes

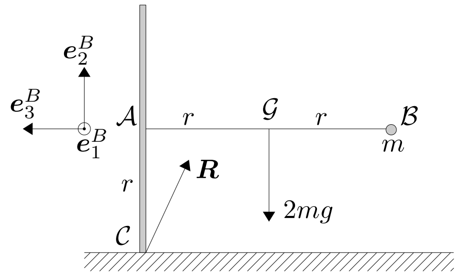
$$(\boldsymbol{K}_1 - \lambda\boldsymbol{M}_1)\,\boldsymbol{a} = \boldsymbol{0}.$$

Since the matrices are already diagonal, we can immediately read off

$$\omega_1^2 = \lambda_1 = \frac{4mg/R}{m} = 4\frac{g}{R}, \quad \omega_2^2 = \lambda_2 = \frac{3mgR/4}{mR^2/16} = 12\frac{g}{R}.$$

Using the symmetry $r \to r$, $\theta \to -\theta$, we could have predicted the mode shapes: pure r oscillation (symmetric) or pure θ oscillation (anti-symmetric).

Problem 3.



Assuming a horizontal $\mathcal{AB}$ axis, we can get to any orientation of the body by first doing a simple rotation about a vertical axis to get the desired direction of $\mathcal{AB}$, followed by a simple rotation about the $\mathcal{AB}$ axis to get the desired contact point on the disc. It follows that the angular velocity of the body is the sum of a vertical part and a part parallel to $\mathcal{AB}$. Thus

$$\boldsymbol{\omega} = \omega_2 \boldsymbol{e}_2^B + \omega_3 \boldsymbol{e}_3^B$$

for some values of ω_2, ω_3. We also note that the angular velocity of the triad B is just the first part:

$$\boldsymbol{\omega}^B = \omega_2 \boldsymbol{e}_2^B.$$

Since the geometric contact point $\mathcal{C}$ is directly below the disc centre $\mathcal{A}$, they have the same velocity. Rolling means that $\mathcal{C}$ is an instantaneous centre of rotation, thus

$$\boldsymbol{v}_\mathcal{C} = \boldsymbol{v}_\mathcal{A} = \boldsymbol{\omega} \times \boldsymbol{r}_{\mathcal{CA}} = (\omega_2 \boldsymbol{e}_2^B + \omega_3 \boldsymbol{e}_3^B) \times r\boldsymbol{e}_2^B = -r\omega_3 \boldsymbol{e}_1^B.$$

The centre of mass $\mathcal{G}$ of the body lies midway between the points $\mathcal{A}$ and $\mathcal{B}$, so the centre of mass velocity, again using rolling, is

$$\boldsymbol{v}_\mathcal{G} = \boldsymbol{\omega} \times \boldsymbol{r}_{\mathcal{CG}} = (\omega_2 \boldsymbol{e}_2^B + \omega_3 \boldsymbol{e}_3^B) \times r(\boldsymbol{e}_2^B - \boldsymbol{e}_3^B) = -r(\omega_2 + \omega_3)\boldsymbol{e}_1^B.$$

We will use the moving point $\mathcal{C}$ as moment point for the angular momentum balance equation, thus

$$\dot{\boldsymbol{L}}_\mathcal{C} + \boldsymbol{v}_\mathcal{C} \times \boldsymbol{p} = \boldsymbol{M}_\mathcal{C}^e.$$

We note that the second term on the left is $\boldsymbol{0}$ since both $\boldsymbol{v}_\mathcal{C}$ and $\boldsymbol{v}_\mathcal{G}$ are parallel to $\boldsymbol{e}_1^B$, and that the moment of force from the contact force $\boldsymbol{R}$ disappears.

Using that $\mathcal{C}$ is an instantaneous centre of rotation, together with the moment of inertia components computed in Problem 4, we have

$$\boldsymbol{L}_\mathcal{C} = mr^2 \left[\left(\frac{17}{4}\omega_2 + 2\omega_3 \right) \boldsymbol{e}_2^B + \left(2\omega_2 + \frac{5}{2}\omega_3 \right) \boldsymbol{e}_3^B \right].$$

Taking time derivatives,

$$\dot{\boldsymbol{L}}_\mathcal{C} = \frac{{}^B d\boldsymbol{L}_\mathcal{C}}{dt} + \boldsymbol{\omega}^B \times \boldsymbol{L}_\mathcal{C}.$$

We get

$$\dot{\boldsymbol{L}}_{\mathcal{C}} = mr^2 \left[\left(\frac{17}{4}\dot{\omega}_2 + 2\dot{\omega}_3 \right) \boldsymbol{e}_2^B + \left(2\dot{\omega}_2 + \frac{5}{2}\dot{\omega}_3 \right) \boldsymbol{e}_3^B \right] + mr^2\omega_2 \left(2\omega_2 + \frac{5}{2}\omega_3 \right) \boldsymbol{e}_1^B.$$

The moment of force about $\mathcal{C}$ comes only from gravity, and is

$$\boldsymbol{M}_{\mathcal{C}}^e = r(\boldsymbol{e}_2^B - \boldsymbol{e}_3^B) \times (-2mg\boldsymbol{e}_2^B) = -2mgr\boldsymbol{e}_1^B.$$

Angular momentum balance now gives

$$\begin{aligned}
\boldsymbol{e}_1^B &: \quad mr^2\omega_2 \left(2\omega_2 + \frac{5}{2}\omega_3 \right) = -2mgr \\
\boldsymbol{e}_2^B &: \quad mr^2 \left(\frac{17}{4}\dot{\omega}_2 + 2\dot{\omega}_3 \right) = 0 \\
\boldsymbol{e}_3^B &: \quad mr^2 \left(2\dot{\omega}_2 + \frac{5}{2}\dot{\omega}_3 \right) = 0.
\end{aligned}$$

The last two equations gives $\dot{\omega}_2 = \dot{\omega}_3 = 0$, so both ω_2 and ω_3 are constants. The first gives

$$\omega_3 = -\frac{4}{5}\omega_2 \left(1 + \frac{g}{r\omega_2^2} \right).$$

The determine the path of the contact point, we note that $\boldsymbol{v}_{\mathcal{C}}$ has constant size, but the direction is rotating with constant angular velocity $\boldsymbol{\omega}^B$. Thus the path must be a circle with radius

$$R = \frac{|\boldsymbol{v}_{\mathcal{C}}|}{|\boldsymbol{\omega}^B|} = \frac{r|\omega_3|}{|\omega_2|} = \frac{4}{5}r \left(1 + \frac{g}{r\omega_2^2} \right).$$

If $\omega_2^2 = 4g/r$, then the radius is $R = r$, which means the centre of mass $\mathcal{G}$ stays fixed, so no horizontal contact force (friction) is required.

Problem 4.

For the disc, since $\boldsymbol{r}_{\mathcal{CA}} = r\boldsymbol{e}_2^B$, we have

$$\boldsymbol{J}_{\mathcal{C}\,\text{disc}}^B = \boldsymbol{J}_{\mathcal{A}\text{disc}}^B + m \begin{bmatrix} r^2 + 0^2 & -0 \cdot r & -0 \cdot 0 \\ -0 \cdot r & 0^2 + 0^2 & -r \cdot 0 \\ -0 \cdot 0 & -r \cdot 0 & 0^2 + r^2 \end{bmatrix} = mr^2 \begin{bmatrix} \frac{1}{4} + 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} + 1 \end{bmatrix}.$$

For the particle, since $\boldsymbol{r}_{\mathcal{CB}} = r\boldsymbol{e}_2^B - 2r\boldsymbol{e}_3^B$, we have

$$\boldsymbol{J}_{\mathcal{C}\,\text{particle}}^B = m \begin{bmatrix} r^2 + (-2r)^2 & -0 \cdot r & -0 \cdot (-2r) \\ -0 \cdot r & 0^2 + (-2r)^2 & -r \cdot (-2r) \\ -0 \cdot (-2r) & -r \cdot (-2r) & 0^2 + r^2 \end{bmatrix} = mr^2 \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

In total

$$\boldsymbol{J}_{\mathcal{C}}^B = mr^2 \begin{bmatrix} \frac{25}{4} & 0 & 0 \\ 0 & \frac{17}{4} & 2 \\ 0 & 2 & \frac{5}{2} \end{bmatrix}.$$

From the zeros in the first row and column (and from the fact that $\boldsymbol{e}_1^B$ is normal to a reflection plane), we see that $\boldsymbol{e}_1^B$ is a principal axis with principal moment of inertia $25mr^2/4$.

The other two principal moments of inertia come form solving an eigenvalue problem for the lower right 2-by-2 matrix. Let $\lambda = J/(mr^2)$. Then

$$\det\left(\begin{bmatrix} \frac{17}{4}-\lambda & 2 \\ 2 & \frac{5}{2}-\lambda \end{bmatrix}\right) = 0 \Leftrightarrow \lambda^2 - \frac{27}{4}\lambda + \frac{53}{8} = 0.$$

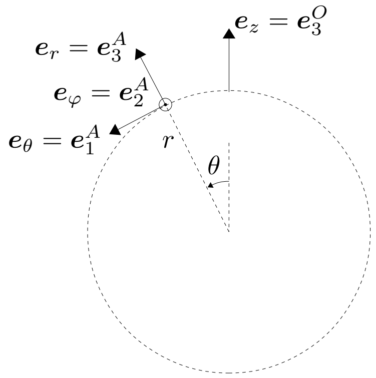
This gives

$$\lambda = \frac{27 \pm \sqrt{305}}{8}.$$

The principal moments of inertia (ordered by size) are

$$\frac{27-\sqrt{305}}{8}mr^2, \quad \frac{27+\sqrt{305}}{8}mr^2, \quad \frac{25}{4}mr^2.$$

Problem 5.



The angular velocity of the first simple rotation is $\dot{\varphi}\boldsymbol{e}_3^O$ and that of the second is $\dot{\theta}\boldsymbol{e}_2^A$, so

$${}^O\boldsymbol{\omega}^A = \dot{\varphi}\boldsymbol{e}_z + \dot{\theta}\boldsymbol{e}_\varphi = \dot{\varphi}[\cos(\theta)\boldsymbol{e}_r - \sin(\theta)\boldsymbol{e}_\theta] + \dot{\theta}\boldsymbol{e}_\varphi.$$

For the velocity we get then get

$$\boldsymbol{v} = \frac{{}^A d\boldsymbol{r}}{dt} + {}^O\boldsymbol{\omega}^A \times \boldsymbol{r} = \dot{r}\boldsymbol{e}_r + \left(\dot{\varphi}[\cos(\theta)\boldsymbol{e}_r - \sin(\theta)\boldsymbol{e}_\theta] + \dot{\theta}\boldsymbol{e}_\varphi\right) \times r\boldsymbol{e}_r = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + r\sin(\theta)\dot{\varphi}\boldsymbol{e}_\varphi.$$

Problem 6. The Routh function $R = L - p_2\dot{q}_2$ depends on q_1 both directly, and indirectly though $\dot{q}_2$. We get

$$\frac{\partial R}{\partial q_1} = \frac{\partial L}{\partial q_1} + \frac{\partial L}{\partial \dot{q}_2}\frac{\partial \dot{q}_2}{\partial q_1} - p_2\frac{\partial \dot{q}_2}{\partial q_1} = \frac{\partial L}{\partial q_1} + \left[\frac{\partial L}{\partial \dot{q}_2} - p_2\right]\frac{\partial \dot{q}_2}{\partial q_1} = \frac{\partial L}{\partial q_1},$$

where the last equality follows from the definition of p_2. The computation for the $\dot{q}_1$ dependence is analogous. (See also "The Theory of Lagrange's Method" section 19, where the same type of computation is done for the Hamilton function).

Since these partial derivatives are equal at any point in time and state, it means that any solution to Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0$$

is also a solution to

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_1}\right) - \frac{\partial R}{\partial q_1} = 0,$$

where p_2 can be treated as a constant.