Matrix Algebra
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Deadline: 2018–04–11, 10.00

Homework # 2

Numbers below refer to problems in Horn, Johnson "Matrix analysis, 2nd ed." A number 1.1.P.2 refers to Problem 2 in Section 1.1.

Note that there are 8 problems in total (see also the back side of the paper sheet).

1. (1.1.P.5) Let $A \in M_n$ be idempotent, that is, $A^2 = A$. Show that each eigenvalue of A is either 0 or 1. Explain why I is the only nonsingular idempotent matrix.

2. (1.1.P.6) Show that all eigenvalues of a nilpotent matrix are 0. Give an example of a nonzero nilpotent matrix. Explain why 0 is the only nilpotent idempotent matrix.

3. (1.2.P21) Let $A \in M_n$ and nonzero vectors $x, v \in \mathbb{C}^n$ be given. Suppose that $c \in \mathbb{C}$, $v^*x = 1$, $Ax = \lambda x$, and the eigenvalues of A are $\lambda, \lambda_2, \ldots, \lambda_n$. Show that the eigenvalues of the "Google" matrix $A(c) = cA + (1-c)\lambda x v^*$ are $\lambda, c\lambda_2, \ldots, c\lambda_n$.

4. (1.3.P1) Let $A, B \in M_n$. Suppose that A and B are diagonalizable and commute. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A and let $\mu_1, \ldots, \mu_n$ be the eigenvalues of B. (a) Show that the eigenvalues of $A + B$ are $\lambda_1 + \mu_{i_1}, \lambda_2 + \mu_{i_2}, \ldots, \lambda_n + \mu_{i_n}$ for some permutation $i_1, \ldots, i_n$ of $1, \ldots, n$. (b) If B is nilpotent, explain why A and $A + B$ have the same eigenvalues. (c) What are the eigenvalues of AB?

5. (1.3.P.7) A matrix $A \in M_n$ is a square root of $B \in M_n$ if $A^2 = B$. Show that every diagonalizable $B \in M_n$ has a square root. Does $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ have a square root? Why?

6. (1.4.P.1) Let nonzero vectors $x, y \in M_n$ be given, let $A = xy^*$ and let $\lambda = y^*x$. Show that

   (a) λ is an eigenvalue of A;

   (b) x is a right and y is a left eigenvector of A associated with λ;

   (c) if $\lambda \neq 0$, then it is the *only* nonzero eigenvalue of A (algebraic multiplicity=1).

   Explain why any vector that is orthogonal to y is in the null space of A. What is the geometric multiplicity of the eigenvalue 0? Explain why A is diagonalizable if and only if $y^*x \neq 0$.

7. (1.4.P.7) In this problem we outline a simple version of the *power method* for finding the largest modulus eigenvalue and an associated eigenvector of $A \in M_n$. Suppose that $A \in M_n$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and that there is exactly one eigenvalue λ_n of maximum modulus $\rho(A)$. If $x^{(0)} \in \mathbb{C}^n$ is *not* orthogonal to a left eigenvector associated with λ_n, show that the sequence

$$x^{(k+1)} = \frac{1}{\sqrt{x^{(k)*}x^{(k)}}} A x^{(k)}, \quad k = 0, 1, 2, \ldots$$

   converges to an eigenvector of A, and the ratios of a given nonzero entry in the vectors $Ax^{(k)}$ and $x^{(k)}$ converge to λ_n.

8. (1.4.P.8) As a continuation of the previous exercise, further eigenvalues (and eigenvectors) of A can be calculated by combining the power method with a *deflation* that delivers a square matrix of size one smaller, whose spectrum (with multiplicities) contains all but one eigenvalue of A. Let $S \in M_n$ be nonsingular and have as its first column an eigenvector $y^{(n)}$ associated with eigenvalue λ_n. Show that $S^{-1}AS = \begin{bmatrix} \lambda_n & * \\ 0 & B \end{bmatrix}$ and the eigenvalues of $B \in M_{n-1}$ are $\lambda_1, \ldots, \lambda_{n-1}$. Another eigenvalue may be calculated from B and the deflation can be repeated until all eigenvalues have been found.