Matrix Algebra
Magnus Jansson / Mats Bengtsson
Deadline: 2018–04–17, 10:00

Homework # 3

Numbers below refer to problems in Horn, Johnson "Matrix analysis." A number 1.1.P.2 refers to Problem 2 in Section 1.1.

In the problems involving Matlab programming: Write a general function that works for any matrix (subject to the limitations specified in the problem). Print and attach to your solutions the code for this function. Illustrate the performance of the function when applied to a matrix of your choice.

1. Show that if A is both upper triangular and unitary, then it is diagonal. What can you say about the diagonal elements (don't forget about the complex valued case)?

2. (2.1.P23) Let $A \in M_n$, let $A = QR$ be a QR factorization, let $R = [r_{ij}]$, and partition A, Q and R according to their columns: $A = [a_1 \dots a_n]$, $Q = [q_1 \dots q_n]$ and $R = [r_1 \dots r_n]$. Explain why $|\det A| = \det R = \prod_{i=1}^{n} r_{ii}$ and why $\|a_i\|_2 = \|r_i\|_2 \geq r_{ii}$ for each $i = 1, \dots, n$, with equality for some i if and only if $a_i = r_{ii} q_i$. Conclude that $|\det A| \leq \prod_{i=1}^{n} \|a_i\|_2$, with equality if and only if either

   (a) some $a_i = 0$ or

   (b) A has orthogonal columns.

   This is *Hadamard's inequality.*

3. Show that a Householder matrix U_w acts as the identity on the subspace $w^{\perp}$ and that it acts as a reflection on the one-dimensional subspace spanned by w; i.e., that $U_w x = x$ if $x \perp w$ and $U_w w = -w$.

4. Assume that a unitary matrix $Q \in M_n$ is given as a product of $d < n$ Household matrices, $Q = U_{w_d} U_{w_{d-1}} \cdots U_{w_1}$.

   a) How many (complex) scalars are needed to store Q in memory, if you use the most space-efficient approach?

   b) Determine a method to calculate the product QA, where $A \in M_{n,k}$, using as few multiplications as possible. How many (complex valued) multiplications are needed? For what values of d can you save computations compared to a standard matrix multiplication algorithm for $n \times n$ times $n \times k$ matrices?

5. (2.2.P1) Solve the following problem and implement the method in Matlab.

   Let $A = [a_{ij}] \in M_n(\mathbb{R})$ be symmetric but *not* diagonal, and choose indices i, j with $i < j$ such that $|a_{ij}| = \max\{|a_{pq}| \; : \; p < q\}$. Define θ by $\cot(2\theta) = (a_{ii} - a_{jj})/2a_{ij}$, let $U(\theta; i, j)$ be the plane rotation (Ex. 2.1.11) (2.2.3 in the old book), and let $B = U(\theta; i, j)^T A U(\theta; i, j) = [b_{pq}]$. Show that $b_{ij} = 0$, $\sum_{p,q=1}^{n} |b_{pq}|^2 = \sum_{p,q=1}^{n} |a_{pq}|^2$, and

$$\sum_{p \neq q} |b_{pq}|^2 < \sum_{p \neq q} |a_{pq}|^2$$

   Explain why a sequence of real orthogonal similarities via plane rotations chosen in this way (at each step, do a plane rotation that annihilates a largest-magnitude off-diagonal entry) converges to a diagonal matrix whose diagonal entries are the eigenvalues of A. How can the corresponding eigenvectors be obtained as a by-product of this process? This is *Jacobi's method* for calculating the eigenvalues of a real symmetric matrix. In practical implementations, it's possible (and preferable) to avoid the calculation of trigonometric functions; see Golub and Van Loan (1996), however this is not necessary in your implementation.

6. Given that a matrix $A \in M_3$ has eigenvalues $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -3$, find a matrix polynomial of the form $r(A) = r_2 A^2 + r_1 A + r_0 I$, such that $r(A) = A^4 + A^{-1}$.

7. (2.4.P2, part) Show that the rank of an upper triangular matrix is at least as large as the number of its nonzero main diagonal entries.

   For an arbitrary $A \in M_n$, show (using Shur's theorem) that the rank of A is not less than the number of nonzero eigenvalues of A. Also, provide an example A where the rank is higher than the number of non-zero eigenvalues.

8. Show that every unitary, Hermitian, and skew-Hermitian matrix is normal. Verify that $A = \begin{bmatrix} 1 & e^{j\pi/4} \\ -e^{j\pi/4} & 1 \end{bmatrix}$ is normal but that no non-zero scalar multiple of A is unitary, Hermitian, or skew-Hermitian.

9. (3.1.P5) Explain why every Jordan block $J_k(\lambda)$ has a one-dimensional eigenspace associated with the eigenvalue λ. Conclude that λ has geometric multiplicity 1 and algebraic multiplicity k, as en eigenvalue of $J_k(\lambda)$.

10. (3.2.P6) The linear transformation $d/dt : p(t) \to p'(t)$ acting on the vector space of all polynomials with degree at most 3 has the basis representation

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   in the basis $B = \{1, t, t^2, t^3\}$. What is the Jordan canonical form of this matrix?