KTH ROYAL INSTITUTE
OF TECHNOLOGY

Lecture 7

- Chapter 6 + Appendix D: Location and perturbation of eigenvalues
- Some other results on perturbed eigenvalue problems
- Chapter 8: Nonnegative matrices

Magnus Jansson/Mats Bengtsson, May, 2018

Eigenvalue Perturbation Results, Motivation

We know from a previous lecture that $\rho(A) \leq |||A|||$ for any *matrix* norm. That is, we know that all eigenvalues are in a circular disk with radius upper bounded by any matrix norm. More precise results?

What can be said about the eigenvalues and eigenvectors of $A + \epsilon B$ when ϵ is small?

Geršgorin circles

Geršgorin's Thm: Let $A = D + B$, where $D = diag(d_1, \dots, d_n)$, and $B = [b_{ij}] \in M_n$ has zeros on the diagonal. Define

$$r_i'(B) = \sum_{\substack{j=1 \\ j \neq i}}^{n} |b_{ij}|$$

$$C_i(D, B) = \{z \in \mathbf{C} : |z - d_i| \leq r_i'(B)\}$$

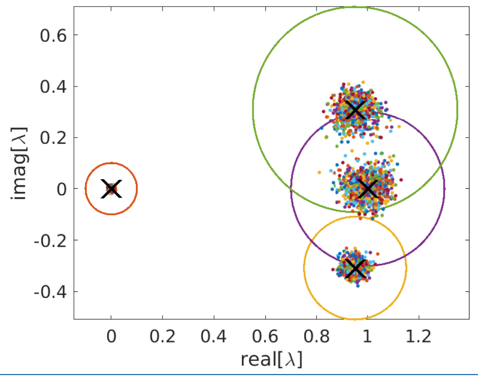
Then, all eigenvalues of A are located in

$$\lambda_k(A) \in G(A) = \bigcup_{i=1}^{n} C_i(D, B) \qquad \forall k$$

The $C_i(D, B)$ are called **Geršgorin circles**.

Geršgorin circles, cont.

If $G(A)$ contains a region of k circles that are disjoint from the rest, then there are k eigenvalues in that region.



Geršgorin, Improvements

Since A^T has the same eigenvalues as A, we can do the same but summing over columns instead of rows. We conclude that

$$\lambda_i(A) \in G(A) \cap G(A^T) \qquad \forall i$$

Since $S^{-1}AS$ has the same eigenvalues as A, the above can be "improved" by

$$\lambda_i(A) \in G(S^{-1}AS) \cap G\big((S^{-1}AS)^T\big) \qquad \forall i$$

for any choice of S. For it to be useful, S should be "simple", e.g., diagonal (see e.g. Corollaries 6.1.6 and 6.1.8).

Invertibility and stability

If $A \in M_n$ is strictly diagonally dominant such that

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad \forall i$$

then

1. A is invertible.
2. If all main diagonal elements are real and positive then all eigenvalues are in the right half plane.
3. If A is Hermitian with all diagonal elements positive, then all eigenvalues are real and positive.

Reducible matrices

A matrix $A \in M_n$ is called *reducible* if

- $n = 1$ and $A = 0$ or
- $n \geq 2$ and there is a permutation matrix $P \in M_n$ such that

  $$P^T A P = \begin{array}{l} \left[\begin{array}{c|c} B & C \\ \hline 0 & D \end{array}\right] \begin{array}{l} \}\ r \\ \}\ n-r \end{array} \\ \ \underbrace{\quad}_{r}\ \underbrace{\quad}_{n-r} \end{array}$$

  for some integer $1 \leq r \leq n-1$.

A matrix $A \in M_n$ that is not reducible is called *irreducible*.
A matrix is irreducible iff it is the adjacency matrix of a *strongly connected* directed graph, "A has the SC property".

Irreducibly diagonally dominant

If $A \in M_n$ is called *irreducibly diagonally dominant* if

i) A is irreducible ($= A$ has the SC property).

ii) A is diagonally dominant,

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad \forall i$$

iii) For at least one row, i,

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

Invertibility and stability, stronger result

If $A \in M_n$ is irreducibly diagonally dominant, then

1. A is invertible.
2. If all main diagonal elements are real and positive then all eigenvalues are in the right half plane.
3. If A is Hermitian with all diagonal elements positive, then all eigenvalues are real and positive.

Perturbation theorems

Thm: Let $A, E \in M_n$ and let A be diagonalizable, $A = S\Lambda S^{-1}$. Further, let $\hat{\lambda}$ be an eigenvalue of $A + E$. Then there is *some* eigenvalue λ_i of A such that

$$|\hat{\lambda} - \lambda_i| \le |||S|||\ |||S^{-1}|||\ |||E||| = \kappa(S)|||E|||$$

for some particular matrix norms (e.g., $|||\cdot|||_1,\ |||\cdot|||_2,\ |||\cdot|||_\infty$).

Cor: If A is a normal matrix, S is unitary $\implies |||S|||_2 = |||S^{-1}|||_2 = 1$. This gives

$$|\hat{\lambda} - \lambda_i| \le |||E|||_2$$

indicating that normal matrices are perfectly conditioned for eigenvalue computations.

Perturbation cont'd

If both A and E are Hermitian, we can use Weyl's theorem (here we assume the eigenvalues are indexed in non-decreasing order):

$$\lambda_1(E) \le \lambda_k(A+E) - \lambda_k(A) \le \lambda_n(E) \qquad \forall k$$

We also have for this case

$$\left[\sum_{k=1}^{n} |\lambda_k(A+E) - \lambda_k(A)|^2\right]^{1/2} \le ||E||_2$$

where $||\cdot||_2$ is the Frobenius norm.

Perturbation of a simple eigenvalue

Let λ be a simple eigenvalue of $A \in M_n$ and let y and x be the corresponding left and right eigenvectors. Then $y^*x \neq 0$.

Thm: Let $A(t) \in M_n$ be differentiable at $t = 0$ and assume λ is a simple eigenvalue of $A(0)$ with left and right eigenvectors y and x. If $\lambda(t)$ is an eigenvalue of $A(t)$ for small t such that $\lambda(0) = \lambda$ then

$$\lambda'(0) = \frac{y^*A'(0)x}{y^*x}$$

Example: $A(t) = A + tE$ gives $\lambda'(0) = \frac{y^*Ex}{y^*x}$.

Perturbation of eigenvalues cont'd

Errors in eigenvalues may also be related to the residual $r = A\hat{x} - \hat{\lambda}\hat{x}$. Assume for example that A is diagonalizable $A = S\Lambda S^{-1}$ and let $\hat{x}$ and $\hat{\lambda}$ be a given complex vector and scalar, respectively. Then there is some eigenvalue of A such that

$$|\hat{\lambda} - \lambda_i| \leq \kappa(S)\frac{||r||}{||\hat{x}||}$$

(for details and conditions see book).
We conclude that a small residual implies a good approximation of the eigenvalue.

Perturbation of eigenvectors with simple eigenvalues

Thm: Let $A(t) \in M_n$ be differentiable at $t = 0$ and assume λ_0 is a simple eigenvalue of $A(0)$ with left and right eigenvectors y_0 and x_0. If $\lambda(t)$ is an eigenvalue of $A(t)$, it has a right eigenvector $x(t)$ for small t normalized such that

$x_0^* x(t) = 1$, with derivative

$$x'(0) = (\lambda_0 I - A(0))^{\dagger}\left(I - \frac{x_0 y_0^*}{y_0^* x_0}\right) A'(0)x_0$$

$B^{\dagger}$ denotes the Moore-Penrose pseudo inverse of a matrix B.

See, e.g., J. R. Magnus and H. Neudecker. *Matrix Differential Calculus with Applications in Statistics and Econometrics.*

Perturbation of eigenvectors with simple eigenvalues: The real symmetric case

Assume that $A \in M_n(\mathbf{R})$ is real symmetric matrix with normalized eigenvectors x_i and eigenvalues λ_i. Further assume that λ_1 is a simple eigenvalue. Let $\hat{A} = A + \epsilon B$ where ϵ is a small scalar, B is real symmetric and let $\hat{x}_1$ be an eigenvector of $\hat{A}$ that approaches x_1 as $\epsilon \to 0$. Then a first order approximation (in ϵ) is

$$\hat{x}_1 - x_1 = \epsilon \sum_{k=2}^{n} \frac{x_k^T B x_1}{\lambda_1 - \lambda_k} x_k + \mathcal{O}(\epsilon^2)$$

Warning: Non-unique derivative in the complex valued case!

Warning, Warning Warning: No extension to multiple eigenvalues!

Literature with perturbation results

J. R. Magnus and H. Neudecker.
Matrix Differential Calculus with Applications in Statistics and Econometrics.
John Wiley & Sons Ltd., 1988, rev. 1999.

H. Krim and P. Forster. *Projections on unstructured subspaces.*
IEEE Trans. SP, 44(10):2634–2637, Oct. 1996.

J. Moro, J. V. Burke, and M. L. Overton.
On the Lidskii-Vishik- Lyusternik perturbation theory for eigenvalues of matrices with arbitrary Jordan structure.
SIAM Journ. Matrix Anal. and Appl., 18(4):793–817, 1997.

F. Rellich. *Perturbation Theory of Eigenvalue Problems.*
Gordon & Breach, 1969.

J. Wilkinson. *The Algebraic Eigenvalue Problem.*
Clarendon Press, 1965.

Chapter 8: Element-wise nonnegative matrices

Def: A matrix $A = [a_{ij}] \in M_{n,r}$ is *nonnegative* if $a_{ij} \geq 0$ for all i, j, and we write this as $A \geq 0$. (Note that this should not be confused with the matrix being nonnegative definite!)
If $a_{ij} > 0$ for all i, j, we say that A is *positive* and write this as $A > 0$. (We write $A > B$ to mean $A - B > 0$ etc.)

We also define $|A| = [\,|a_{ij}|\,]$.

Typical applications are problems in which matrices have elements corresponding to

- probabilities (e.g., Markov chains)
- power levels or power gain factors (e.g., in power control for wireless systems).
- Non-negative weights/costs in graphs.

Nonnegative matrices: Some properties

Let $A, B \in M_n$ and $x \in \mathbf{C}^n$. Then

- $|Ax| \leq |A||x|$
- $|AB| \leq |A||B|$
- If $A \geq 0$, then $A^m \geq 0$; if $A > 0$, then $A^m > 0$.
- If $A \geq 0$, $x > 0$, and $Ax = 0$ then $A = 0$.
- If $|A| \leq |B|$, then $\|A\| \leq \|B\|$, for any absolute norm $\|\cdot\|$; that is, a norm for which $\|A\| = \|\,|A|\,\|$.

Nonnegative matrices: Spectral radius

Lemma: If $A \in M_n$, $A \geq 0$, and if the row sums of A are constant, then $\rho(A) = |||A|||_\infty$. If the column sums are constant, then $\rho(A) = |||A|||_1$.

The following theorem can be used to give upper and lower bounds on the spectral radius of **arbitrary** matrices.

Thm: Let $A, B \in M_n$. If $|A| \leq B$, then $\rho(A) \leq \rho(|A|) \leq \rho(B)$.

Nonnegative matrices: Spectral radius

Thm: Let $A \in M_n$ and $A \geq 0$. Then

$$\min_i \sum_{j=1}^{n} a_{ij} \leq \rho(A) \leq \max_i \sum_{j=1}^{n} a_{ij}$$

$$\min_j \sum_{i=1}^{n} a_{ij} \leq \rho(A) \leq \max_j \sum_{i=1}^{n} a_{ij}$$

Thm: Let $A \in M_n$ and $A \geq 0$. If $Ax = \lambda x$ and $x > 0$, then $\lambda = \rho(A)$.

Positive matrices

For positive matrices we can say a little more.

Perron's theorem: If $A \in M_n$ and $A > 0$, then

1. $\rho(A) > 0$
2. $\rho(A)$ is an eigenvalue of A
3. There is an $x \in \mathbf{R}^n$ with $x > 0$ such that $Ax = \rho(A)x$
4. $\rho(A)$ is an algebraically (and geometrically) simple eigenvalue of A
5. $|\lambda| < \rho(A)$ for every eigenvalue $\lambda \neq \rho(A)$ of A
6. $[A/\rho(A)]^m \to L$ as $m \to \infty$, where $L = xy^T$, $Ax = \rho(A)x$, $y^TA = \rho(A)y^T$, $x > 0$, $y > 0$, and $x^Ty = 1$.

$\rho(A)$ is sometimes called a Perron root and the vector $x = [x_i]$ a Perron vector if it is scaled such that $\sum_{i=1}^n x_i = 1$.

Nonnegative matrices

Generalization of Perron's theorem to general non-negative matrices?

Thm: If $A \in M_n$ and $A \geq 0$, then

1. $\rho(A)$ is an eigenvalue of A
2. There is a non-zero $x \in \mathbf{R}^n$ with $x \geq 0$ such that $Ax = \rho(A)x$

For stronger results, we need a stronger assumption on A.

Irreducible matrices

Reminder: A matrix $A \in M_n$, $n \geq 2$ is called *reducible* if there is a permutation matrix $P \in M_n$ such that

$$P^TAP = \begin{array}{ccl} \left[\begin{array}{c|c} B & C \\ \hline 0 & D \end{array}\right] & \begin{array}{l} \}\ r \\ \}\ n-r \end{array} \\ \underbrace{\qquad}_{r}\underbrace{\qquad}_{n-r} & \end{array}$$

for some integer $1 \leq r \leq n-1$.

A matrix $A \in M_n$ that is not reducible is called *irreducible*.

Thm: A matrix $A \in M_n$ with $A \geq 0$ is irreducible iff $(I + A)^{n-1} > 0$

Irreducible matrices

Frobenius' theorem: If $A \in M_n$, $A \geq 0$ is irreducible, then

1. $\rho(A) > 0$
2. $\rho(A)$ is an eigenvalue of A
3. There is an $x \in \mathbf{R}^n$ with $x > 0$ such that $Ax = \rho(A)x$
4. $\rho(A)$ is an algebraically (and geometrically) simple eigenvalue of A
5. If there are exactly k eigenvalues with $|\lambda_p| = \rho(A)$, $p = 1, \ldots, k$, then
   - $\lambda_p = \rho(A)e^{i2\pi p/k}$, $p = 0, 1, \ldots, k-1$ (suitably ordered)
   - If λ is any eigenvalue of A, then $\lambda e^{i2\pi p/k}$ is also an eigenvalue of A for all $p = 0, 1, \ldots, k-1$
   - $\mathrm{diag}[A^m] \equiv 0$ for all m that are not multiples of k (e.g. $m = 1$).

Primitive matrices

A matrix $A \in M_n$, $A \geq 0$ is called *primitive* if

- A is irreducible
- $\rho(A)$ is the only eigenvalue with $|\lambda_p| = \rho(A)$.

Thm: If $A \in M_n$, $A \geq 0$ is primitive, then

$$\lim_{m \to \infty} [A/\rho(A)]^m = L$$

where $L = xy^T$, $Ax = \rho(A)x$, $y^TA = \rho(A)y^T$, $x > 0$, $y > 0$, and $x^Ty = 1$.

Thm: If $A \in M_n$, $A \geq 0$, then it is primitive iff $A^m > 0$ for some $m \geq 1$.

Stochastic matrices

A nonnegative matrix with all its row sums equal to 1 is called a (row) stochastic matrix.

A column stochastic matrix is the transpose of a row stochastic matrix.

If a matrix is both row and column stochastic it is called doubly stochastic.

Stochastic matrices cont'd

- The set of stochastic matrices in M_n is compact and convex.
- Let $\mathbf{1} = [1, 1, \ldots, 1]^T$. A matrix is stochastic if and only if $A\mathbf{1} = \mathbf{1} \implies \mathbf{1}$ is an eigenvector with eigenvalue $+1$, for all stochastic matrices.
- An example of a doubly stochastic matrix is $A = [|u_{ij}|^2]$ where $U = [u_{ij}]$ is a unitary matrix. Also, notice that all permutation matrices are doubly stochastic.

Thm: A matrix is doubly stochastic if and only if it can be written as a convex combination of a finite number of permutation matrices.

Corr: The maximum of a convex function on the set of doubly stochastic matrices is attained at a permutation matrix!

Example, Markov processes

Consider a discrete stochastic process that at each time instant is in one of the states $S_1, \ldots, S_n$. Let p_{ij} be the probability to change from state S_i to state S_j. Note that the transition matrix $P = [p_{ij}]$, is a stochastic matrix.
Let $\mu_i(t)$ denote the probability of being in state S_i at time t and $\mu(t) = [\mu_1(t), \ldots, \mu_n(t)]$, then $\mu(t+1) = \mu(t)P$.

If P is primitive (other terms are used in the statistics literature), then $\mu(t) \to \mu^\infty$ as $t \to \infty$ where $\mu^\infty = \mu^\infty P$, no matter what $\mu(0)$ is. μ^∞ is called the stationary distribution.

Nice overview article: S. U. Pillai, T. Suel, S. Cha, *The Perron Frobenius Theorem: Some of its applications,*
IEEE Signal Processing Magazine, Mar. 2005.

Further results

Other books contain more results.
In "Matrix Theory", vol. II by Gantmacher, for example, you can find results such as:

Thm: If $A \in M_n$, $A \geq 0$ is irreducible, then

$$(\alpha I - A)^{-1} > 0$$

for all $\alpha > \rho(A)$.

(Useful, for example, in connection with power control of wireless systems).