Rigid Body Dynamics (SG2150)
Solution to Exam, 2018-10-25, 08.00-13.00

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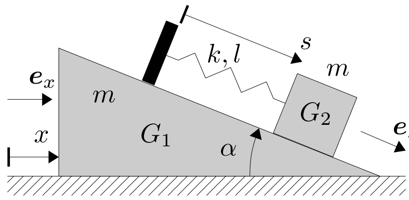
Problem 1.



There is no rotation, so the kinetic energy is

$$T = \frac{1}{2} m \left|\boldsymbol{v}_{G_1}\right|^2 + \frac{1}{2} m \left|\boldsymbol{v}_{G_2}\right|^2 .$$

The velocites are

$$\boldsymbol{v}_{G_1} = \dot{x}\boldsymbol{e}_x \text{ and } \boldsymbol{v}_{G_2} = \dot{x}\boldsymbol{e}_x + \dot{s}\boldsymbol{e}_s.$$

Using $\boldsymbol{e}_x \bullet \boldsymbol{e}_s = c_\alpha$, we get

$$T = m\left[\dot{x}^2 + c_\alpha \dot{x}\dot{s} + \frac{1}{2}\dot{s}^2\right].$$

Only consevative forces do work, namely the spring force and the gravity force on the upper mass. The potential energy becomes

$$V = -mgs_\alpha s + \frac{k}{2}\left(s - l\right)^2$$

The system is a conservative system with two degrees of freedom x and s, and with Lagrange function

$$L = T - V = m\left[\dot{x}^2 + c_\alpha \dot{x}\dot{s} + \frac{1}{2}\dot{s}^2\right] + mgs_\alpha s - \frac{k}{2}\left(s - l\right)^2$$

To get Lagrange's equations we compute

$$\begin{aligned} p_x = \frac{\partial L}{\partial \dot{x}} &= m\left[2\dot{x} + c_\alpha \dot{s}\right], \\ \frac{\partial L}{\partial x} &= 0, \\ p_s = \frac{\partial L}{\partial \dot{s}} &= m\left[c_\alpha \dot{x} + \dot{s}\right], \\ \frac{\partial L}{\partial s} &= mgs_\alpha - k\left(s - l\right). \end{aligned}$$

Lagrange's equations are

$$\dot{p}_x - \frac{\partial L}{\partial x} = m\left[2\ddot{x} + c_\alpha \ddot{s}\right] = 0,$$
$$\dot{p}_s - \frac{\partial L}{\partial s} = m\left[c_\alpha \ddot{x} + \ddot{s}\right] - mgs_\alpha + k\left(s - l\right) = 0.$$

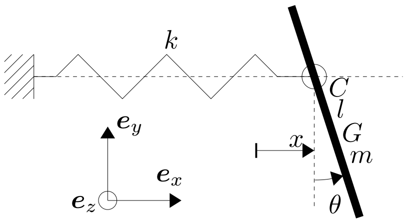
At the initial time point we get

$$m\left[2\ddot{x}(0) + c_\alpha \ddot{s}(0)\right] = 0,$$
$$m\left[c_\alpha \ddot{x}(0) + \ddot{s}(0)\right] = mgs_\alpha - k\left(s(0) - l\right) = mgs_\alpha - kl,$$

which is solved to give

$$\ddot{x}(0) = \frac{c_\alpha}{2 - c_\alpha^2}\left(\frac{kl}{mg} - s_\alpha\right) g,$$
$$\ddot{s}(0) = -\frac{2}{2 - c_\alpha^2}\left(\frac{kl}{mg} - s_\alpha\right) g$$

Problem 2.



Use the coordinate x of the point C along the line, with $x = 0$ corresponding to an unstressed spring, and the angle θ as generalised coordinates.

To find equilibrium points, we study the potential energy. The gravity force and the spring force contributes. The centre of mass G of the rod has distance l from C. We find

$$V = -mgl\cos(\theta) + \frac{k}{2}x^2 = \frac{mg}{l}\left[-l^2\cos(\theta) + \frac{1}{2}x^2\right].$$

using $k = mg/l$. Equilibrium points are found from

$$\frac{\partial V}{\partial x} = \frac{mg}{l}x = 0$$
$$\frac{\partial V}{\partial \theta} = mgl\sin(\theta) = 0.$$

The first equation gives x=0. The second equation gives $\sin(\theta) = 0$. We find

$$x_1 = 0,\ \theta_1 = 0,\ \text{or } x_2 = 0,\ \theta_2 = \pi$$

as the two equilibrium points.

At the second equilibrium point

$$\frac{\partial^2 V}{\partial \theta^2} = mgl\cos(\theta_2) = -mgl < 0,$$

so the stiffness matrix $\boldsymbol{K}_2$ is not positive definite, and equilibrium point 2 is not stable.

At the first equilibrium point, the stiffness matrix is

$$\boldsymbol{K}_1 = \frac{mg}{l}\begin{bmatrix} 1 & 0 \\ 0 & l^2 \end{bmatrix}.$$

To compute the kinetic energy, we will use the "two parts" formula, since there is no obvious instantaneous centre of rotation. Thus we need the velocity of the centre of mass G and the angular velocity $\boldsymbol{\omega}$ of the rod. The angular velocity is $\boldsymbol{\omega} = \dot{\theta}\boldsymbol{e}_z$. The velocity of the point C is $\dot{x}\boldsymbol{e}_x$. As we are considering small oscillations about the equilibrium point (x_1, θ_1) it is enough to compute the velocity in the equilibrium point configuration, where the rod is vertical. The velocity connection formula then gives

$$\boldsymbol{v}_G = \boldsymbol{v}_A + \boldsymbol{\omega} \times \boldsymbol{r}_{CG} = \left(\dot{x} + l\dot{\theta}\right)\boldsymbol{e}_x.$$

This gives a total kinetic energy of

$$T_1 = \frac{1}{2}m|\boldsymbol{v}_G|^2 + \frac{1}{2}J_{Gzz}\dot{\theta}^2 = \frac{1}{2}m\left(\dot{x} + l\dot{\theta}\right)^2 + \frac{1}{2}\frac{m(6l)^2}{12}\dot{\theta}^2 = m\left[\frac{1}{2}\dot{x}^2 + l\dot{x}\dot{\theta} + 2l^2\dot{\theta}^2\right].$$

This gives the mass matrix

$$\boldsymbol{M}_1 = m\begin{bmatrix} 1 & l \\ l & 4l^2 \end{bmatrix}.$$

The eigenvalue problem now becomes

$$(\boldsymbol{K}_1 - \lambda\boldsymbol{M}_1)\,\boldsymbol{a} = \boldsymbol{0}.$$

With $\lambda = g\beta/l$ we get

$$\frac{mg}{l}\begin{bmatrix} 1-\beta & -l\beta \\ -l\beta & l^2(1-4\beta) \end{bmatrix}\boldsymbol{a} = \boldsymbol{0}.$$

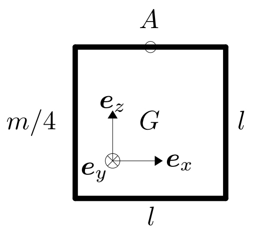
The matrix is singular when the determinant is zero, which gives

$$(1-\beta)(1-4\beta) - \beta^2 = 0 \Rightarrow \beta = \frac{5 \pm \sqrt{13}}{6}.$$

As expected, both β values are positive, confirming that the equilibrium point is stable. We get the answer

$$\omega_1^2 = \lambda_1 = \frac{5-\sqrt{13}}{6}\frac{g}{l}, \quad \omega_2^2 = \lambda_2 = \frac{5+\sqrt{13}}{6}\frac{g}{l}.$$

Problem 3.



Since both the x-z and the y-z planes through the point A are planes of mirror symmetry for the frame, the x, y, and z axes through A are principal axes.

Let G be the centre of mass for the frame. For one of the rods, we get

$$J_{Gyy,\text{rod}} = \frac{m}{4}\frac{l^2}{12} + \frac{m}{4}\left(\frac{l}{2}\right)^2 = \frac{ml^2}{12}.$$

In total, we get

$$J_{Gyy} = 4\frac{ml^2}{12} = \frac{ml^2}{3}.$$

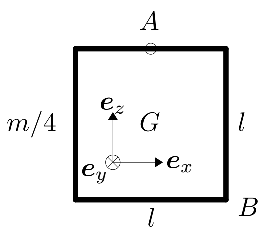
Since G is a point of four-fold rotation symmetry about the y axis, $J_{Gxx} = J_{Gzz}$ and since the frame is planar, $J_{Gyy} = J_{Gxx} + J_{Gzz}$. This gives

$$J_{Gxx} = J_{Gzz} = \frac{1}{2}J_{Gyy} = \frac{ml^2}{6}.$$

Changing the moment point to A finally gives

$$J_{Axx} = J_{Gxx} + m\left(\frac{l}{2}\right)^2 = \frac{5}{12}ml^2,$$
$$J_{Ayy} = J_{Gyy} + m\left(\frac{l}{2}\right)^2 = \frac{7}{12}ml^2,$$
$$J_{Azz} = J_{Gzz} = \frac{1}{6}ml^2.$$

Problem 4.



Consider first the particle. If we denote the impulse on the frame from the particle as $\boldsymbol{S}$, the impulse on the particle becomes $-\boldsymbol{S}$. Momentum balance for the particle gives

$$-\boldsymbol{S} = \boldsymbol{p}_\text{f} - \boldsymbol{p}_\text{i} = \left(\frac{29}{49} - 1\right) mv_0\boldsymbol{e}_y = -\frac{20}{49}mv_0\boldsymbol{e}_y \Rightarrow \boldsymbol{S} = \frac{20}{49}mv_0\boldsymbol{e}_y.$$

Next consider the frame. Using the point A as the moemnt point gets rid of the constraint impulse at A. Angular momentum balance about A gives

$$\boldsymbol{J}_A\left(\boldsymbol{\omega}_\text{f} - \boldsymbol{0}\right) = \boldsymbol{r}_{AB} \times \boldsymbol{S}$$

where

$$\boldsymbol{r}_{AB} = l\left(\frac{1}{2}\boldsymbol{e}_x - \boldsymbol{e}_z\right)$$

Using the moments of inertia from Problem 3, and denoting $\boldsymbol{\omega}_{\mathrm{f}} = \omega_x \boldsymbol{e}_x + \omega_y \boldsymbol{e}_y + \omega_z \boldsymbol{e}_z$, we get the component equations

$$\frac{5}{12}ml^2\omega_x = \frac{20}{49}mv_0 l,$$
$$\frac{7}{12}ml^2\omega_y = 0,$$
$$\frac{1}{6}ml^2\omega_z = \frac{10}{49}mv_0 l,$$

which gives

$$\boldsymbol{\omega}_{\mathrm{f}} = \frac{12}{49}\left(4\boldsymbol{e}_x + 5\boldsymbol{e}_z\right)\frac{v_0}{l}.$$

Using this angular velocity and that A is an instananeous (actually permanent) centre of rotation, we can compute the velocity of the point B:

$$\boldsymbol{v}_{B,\mathrm{f}} = \boldsymbol{0} + \boldsymbol{\omega}_{\mathrm{f}} \times \boldsymbol{r}_{AB} = \frac{78}{49}v_0\boldsymbol{e}_y.$$

The coefficient of restitution e can now be computed from the relative velocities in the normal (y) direction

$$\frac{78}{49}v_0 - \frac{29}{49}v_0 = -e\left(0 - v_0\right) \Rightarrow e = 1,$$

so the collision is elastic.

As a check of this, we can compute the kinetic energy after impact

$$T_{\mathrm{f}} = \frac{1}{2}m\left(\frac{29}{49}v_0\right)^2 + \frac{1}{2}\left[\frac{5}{12}ml^2\left(\frac{48}{49}\frac{v_0}{l}\right)^2 + \frac{1}{6}ml^2\left(\frac{60}{49}\frac{v_0}{l}\right)^2\right] = \frac{1}{2}mv_0^2 = T_{\mathrm{i}}.$$

Problem 5. A cyclic coordinate q_i gives the conserved quantity

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

For derivations of this equation and the other question, see section 19 in *The Theory of Lagrange's Method.*

Problem 6. First we note that q_1 is a cyclic coordinate, so

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = 2\dot{q}_1 + c_\alpha \dot{q}_2 = \text{constant} = 0,$$

the last equality by the initial conditions. This can be integrated in time to give

$$2q_1 + c_\alpha q_2 = \text{constant} = 2c_\alpha,$$

again by the inital conditions. It is clear that (assuming $c_\alpha > 0$) a maximum in $q_2(t)$ gives a simultaneous minimum in $q_1(t)$ and vice versa.

Secondly note that the Lagrange function does not explicitly depend on time. Thus

$$\dot{q}_1 p_1 + \dot{q}_2 p_2 - L = \dot{q}_1^2 + c_\alpha \dot{q}_1 \dot{q}_2 + \frac{1}{2}\dot{q}_2^2 - s_\alpha q_2 + \frac{1}{2}\left(q_2 - 1\right)^2 = \text{constant} = \frac{1}{2} - 2s_\alpha,$$

again using the initial conditions. At a maximum or minimum of $q_2(t)$, the time derivative $\dot{q}_2(t)$ must be zero, and from $p_1 = 0$ so must $\dot{q}_1(t)$. Thus the maxima and minima of q_2 are thus given by

$$-s_\alpha q_2 + \frac{1}{2}(q_2 - 1)^2 = \frac{1}{2} - 2s_\alpha$$

with roots

$$q_{2,\max} = 2, \quad q_{2,\min} = 2s_\alpha.$$

The corresponding q_1 values are then (assuming $c_\alpha > 0$)

$$q_{1,\min} = 0, \quad q_{1,\max} = c_\alpha (1 - s_\alpha).$$

(If $c_\alpha < 0$ the q_1 max and min values are exchanged).

Note: The Lagrange function actually comes from Problem 1 with $k = mg/l$.