Hermite-Fourier spectral method for Vlasov-Poisson systems
Impact of different collisional operators

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Introduction

A Plasma is an ionized gas and consists of electrons and at least one type of ions. The kinetic description of it enables the study of various phenomena, which are not accessible with a fluid approach.
The Vlasov equation describes the time evolution of the distribution function in phase space. Combining it with the Maxwell equations leads to self-consistent description. In the electrostatic limit, the system for *1D1V* is:

$$\frac{\partial f_s}{\partial t} + v\frac{\partial f_s}{\partial x} + \frac{q_s}{m_s}E\frac{\partial f_s}{\partial v} = 0$$

$$\frac{\partial E}{\partial x} = \sum_s q_s n_s \qquad n_s = \int f_s(x,v,t)\,dv$$

The distribution function can be expanded in Fourier (for the spatial part) and Hermite (for the velocity part) basis functions leading to a system of ODEs. It allows a high accuracy and offers with Crank-Nicolson time discretization exact conservation of total mass, momentum and energy.

Method

Expansion with asymmetrically weighted Hermite functions (n=0: Maxwellian):

$$\Psi_n(\xi) = (\pi 2^n n!)^{-1/2} H_n(\xi) e^{-\xi^2} \qquad \xi_s = \frac{(v-u_s)}{\alpha_s}$$
$$\Psi^n(\xi) = (2^n n!)^{-1/2} H_n(\xi)$$

and Fourier functions: $\Phi_k(x) = \exp\left(\frac{2\pi i k x}{L}\right)$

The distribution function is:

$$f_s(x,v,t) = \sum_{m=0}^{N_H-1}\sum_{k=-N}^{N} C^s_{m,k}(t)\Psi_m(\xi_s)\Phi_k(x)$$

the system of ODEs:

$$\frac{\mathrm{d}C^s_{m,n}}{\mathrm{d}t} + \alpha_s\frac{2\pi i n}{L}\left(\sqrt{\frac{m+1}{2}}C^s_{m+1,n} + \sqrt{\frac{m}{2}}C^s_{m-1,n} + \frac{u_s}{\alpha_s}C^s_{m,n}\right) - \frac{q_s}{m_s}\frac{\sqrt{2m}}{\alpha_s}\sum_{j=-N}^{N} E_{n-j}C_{m-1,j} = 0.$$

and the electric field: $E_k = -\frac{iL}{2\pi k}\sum_s q_s\alpha_s C^s_{0,k}\ ,\ 0 \text{ if } k=0$

The system of ODEs is solved by a Newton-Krylov solver by minimizing the residual:

$$\boldsymbol{G}(\boldsymbol{C}^{t+1},\boldsymbol{E}^{t+1}) = \boldsymbol{C}^{t+1} - \boldsymbol{C}^{t} - \Delta t[\mathbb{L}_1\boldsymbol{C}^{t+1/2} + \mathcal{N}(\boldsymbol{C}^{t+1/2},\boldsymbol{E}^{t+1/2})] = 0$$

with termination criterion: $\|\boldsymbol{G}(\boldsymbol{C}^N,\boldsymbol{E}^N)\|_2 \le \tau_a + \tau_r\|\boldsymbol{G}(\boldsymbol{C}^0,\boldsymbol{E}^0)\|_2$

The linear Newton step $\mathbb{J}_n\delta\binom{\boldsymbol{C}^N}{\boldsymbol{E}^N} = -\boldsymbol{G}(\boldsymbol{C}^N,\boldsymbol{E}^N)$ is solved with GMRES where the Jacobian is approximated by a directional derivative. Preconditioning is used with:

$$\left[\mathbb{I} - \frac{\Delta t}{2}\mathbb{L}_1 - \frac{\Delta t}{2}\mathcal{N}(\boldsymbol{C}^0,\boldsymbol{E}^0)\right]^{-1}$$

Purpose

The particle streaming term $v\partial f/\partial x$ causes infinitesimally small velocity structures. The truncation of C_m for $m>N_H$ is like a hard-wall boundary, resulting in a backwards flow of free energy and recurrence of the initial state after t_{rec} ($\sim\sqrt{N_H}$). Therefore, a collision term or a filter is needed. The behaviour of different operators with varying N_H is investigated.

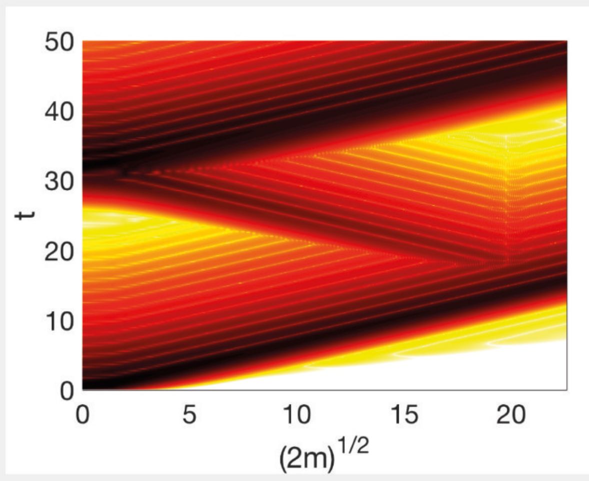
1) Lenard-Bernstein: $\nu\frac{m(m-1)(m-2)}{(N_H-1)(N_H-2)(N_H-3)}C_{m,k}$
m, p and W preserved.
2) Hyper-Collisions: $m^l\nu C_{m,k}$ $\qquad \nu = 1/(\Delta t N_H^l)$
ν is set that the energy dissipates in Δt.
3) Hou-Li filter: $\exp\left(-36(m/(N_H-1))^{36}\right)$
4) Dougherty:
$$\nu_0\frac{n}{T^{3/2}}\left[\left(\frac{2T}{\alpha_s^2}-1\right)\sqrt{(m+3)(m+4)}C_{m+2,k} - mC_{m,k} + \frac{v_{avg}}{\alpha_s}\sqrt{2(m+2)}C_{m+1,k}\right]$$

The change in W_E is equal to a flux between C_0 and C_1. With a transformation, one can depict the flow of energy as a forward and backward hermite flux travelling along characteristics. Three testing cases were investigated with the initial distribution function:

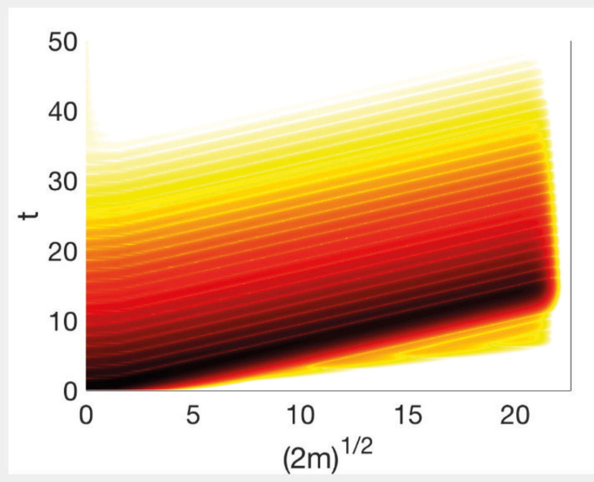
$$f_e(x,v) = \frac{n_0\left[1+\epsilon\sin(2\pi x/L)\right]}{\sqrt{\pi}\alpha_e}\exp\left[-\left(\frac{v-u_e}{\alpha_e}\right)^2\right]$$

Results

Linear Landau Damping
non-dissipative damping ~ exp ($-\gamma t$) of longitudinal electrostatic waves with infinitesimal amplitude

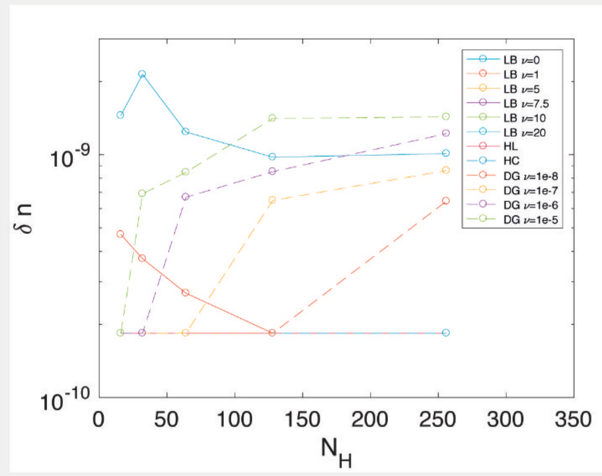


a) without damping

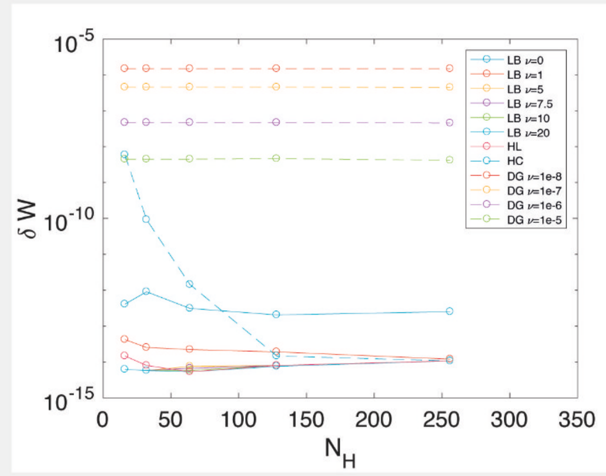


b) Hou-Li filtering

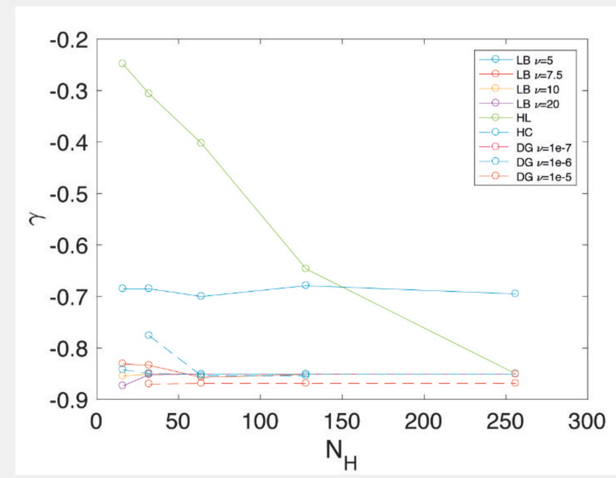
Flow of free energy depicted by Hermite flux.
The spectra clear up by introducing a collision operator.



δn
HC and HL yield best results



δW
HC converges very fast
HL,LB are best

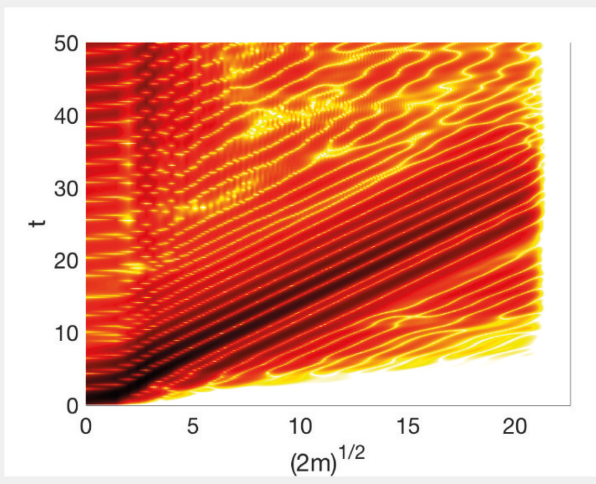


γ
HL converges quickly
HC yields good results
LB depends on choice of ν

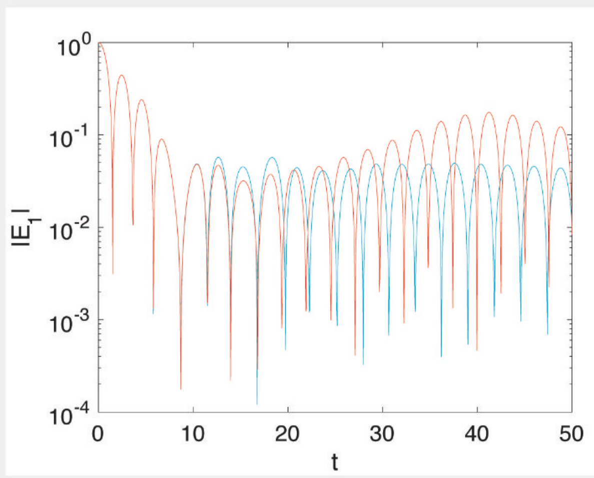
Runtime
LB,HL,HC ~ N_H
DG ~ $N_H^{1.5}$

dynamic N_H:
δn ~10^{-10} for HC,HL,LB
δW ~10^{-15} for LB,HL
δW ~10^{-9} for HC

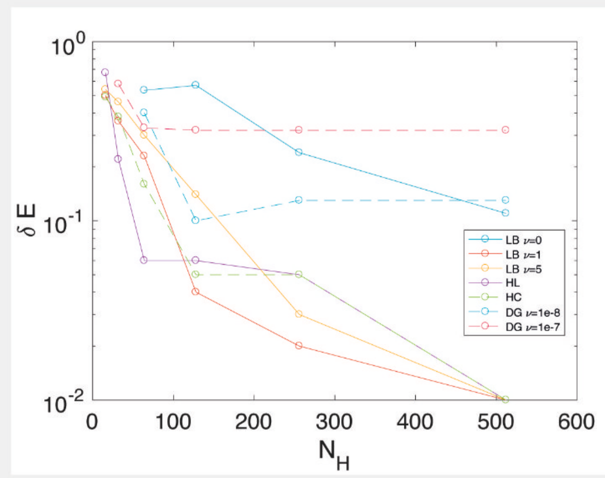
Nonlinear Landau Damping
higher initial perturbation, leading to growth of the electrical field after initial decay



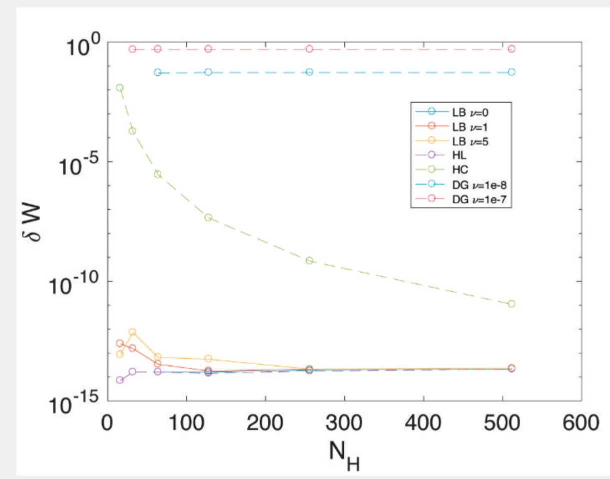
strong non-linearity introduces phase mixing



$E_1(t)$ with reference solution



δE
HC and HL converge quickly
LB depends on choice of ν

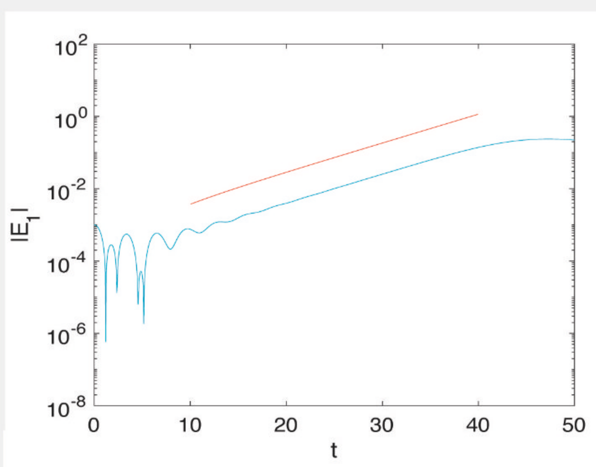


δW
HL,LB yield good results
HC converges quickly

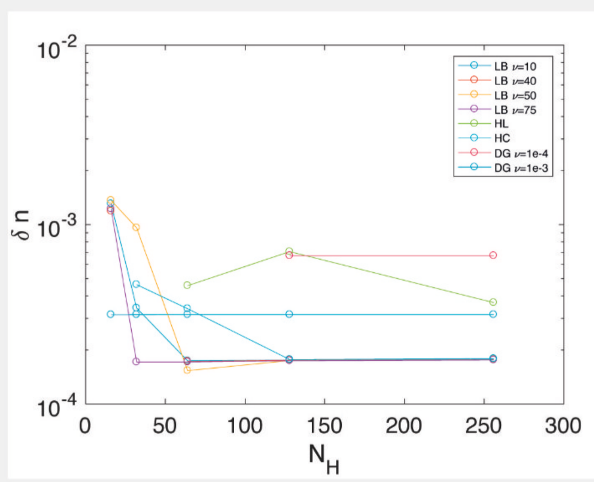
Runtime
LB,HL,HC ~ N_H
DG ~ $N_H^{1.5}$

Error δE defined as: $\frac{\int_0^{\omega_{pe}T}|E_x^{1,0,0}(t) - E_x^{1,0,0,ref}(t)|dt}{\int_0^{\omega_{pe}T}|E_x^{1,0,0,ref}(t)|dt}$.

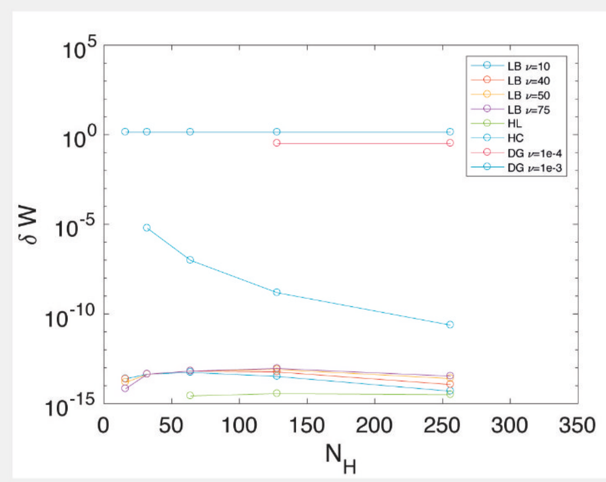
Two Stream Instability
two beams with opposite drift velocities leading to plasma wave excitation



$E_1(t)$ with fit



δn: all operators yield good results



δW: HL, LB have best results
HC poor, DG worst

δγ < 0.11
δn ~10^{-4} if convergence

Runtime
LB,HL,NC ~ N_H
DG ~ $N_H^{1.5}$

Conclusions

It could be shown, that a Hyper-Collisions operator or a Hou-Li filter are attractive alternatives to the Lenard-Bernstein operator regarding the conservation quantities and the expected solution. Both do not have to be tuned regarding the test case in contrast to the Lenard-Bernstein operator and improve quickly for higher N_H. The implementation in a dynamic algorithm is also possible. The Dougherty operator yields poor results for all test cases.

References

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[5] Kelley, C. T. "Iterative methods for linear and nonlinear equations, SIAM, Philadelphia, 1995." MR 96d 65002. [6] Pezzi, Oreste, et al. "Eulerian simulations of collisional effects on electrostatic plasma waves." Physics of Plasmas (1994-present) 20.9 (2013): 092111.