SF2812 Applied linear optimization, final exam
Thursday March 17 2016 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the linear programming problem (LP) defined as

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 3 & 1 & -1 & 0 \\ 2 & 2 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 12 \\ 16 \end{pmatrix},$$
$$c = \begin{pmatrix} -1 & 1 & 2 & -1 \end{pmatrix}^T.$$

A friend of yours claims that she has computed an optimal solution $\widehat{x} = (1\ 9\ 0\ 4)^T$ by an interior method. However, she did not take note of the corresponding dual solution, so she is now unable to verify optimality.

Help your friend by first showing that $\widehat{x}$ can be written as a convex combination of two basic feasible solutions. Then, verify optimality of these basic feasible solutions and finally show that $\widehat{x}$ is optimal. ....(10p)

Hint: Your may find one or several of the results below useful.

$$\begin{pmatrix} 3 & 1 & -1 & 0 \\ 2 & 2 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & -1 & 0 \\ 2 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & -1 & 0 \\ 2 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & -1 & 0 \\ 2 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

2. Consider the linear program

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & 3 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad c = \begin{pmatrix} 2 & -1 & 0 & 2 \end{pmatrix}^T.$$

Assume that we want to solve (LP) using a primal-dual interior method. Assume further that we initially choose $x^{(0)} = (1\ 1\ 1\ 1)^T$, $y^{(0)} = (1\ -1)^T$, $s^{(0)} = (1\ 1\ 1\ 1)^T$, and $\mu = 0.1$. Here, y and s denote the dual variables.

(a) Formulate the linear system of equations to be solved in the first iteration of the primal-dual interior method for the given initial values. First formulate the general form and then add explicit numerical values into the system of equations. .... (7p)

(b) The solution to the above system of linear equations is given by

$$\Delta x \approx \begin{pmatrix} -0.4171 \\ -0.2195 \\ -1.0537 \\ -1.0756 \end{pmatrix}, \quad \Delta y \approx \begin{pmatrix} 0.1537 \\ 0.1756 \end{pmatrix}, \quad \Delta s \approx \begin{pmatrix} -0.4829 \\ -0.6805 \\ 0.1537 \\ 0.1756 \end{pmatrix}.$$

Show how these values may be used to determine $x^{(1)}$, $y^{(1)}$ and $s^{(1)}$ in a suitable way. Complete the calculations up to the point where you would need a calculator. .... (3p)

3. Consider the integer program (IP) defined by

$$(IP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \\ & Cx \geq d, \\ & x \geq 0, \quad x \text{ integer}. \end{array}$$

Let z_{IP} denote the optimal value of (IP).

Associated with (IP) we may define the dual problem (D) as

$$(D) \qquad \begin{array}{ll} \text{maximize} & \varphi(u) \\ \text{subject to} & u \geq 0, \end{array}$$

where $\varphi(u) = \min\{c^T x + u^T(b - Ax) : Cx \geq d, x \geq 0 \text{ integer}\}$. Let z_D denote the optimal value of (D).

Let (LP) denote the linear program obtained from (IP) by relaxing the integer requirement, i.e.,

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \\ & Cx \geq d, \\ & x \geq 0. \end{array}$$

Let z_{LP} denote the optimal value of (LP).

Show that $z_{IP} \geq z_D \geq z_{LP}$. .... (10p)

4. Consider the linear program (LP) given by

$$(LP) \quad \begin{array}{ll} \text{minimize} & 2x_1 + 2x_2 + x_3 \\ \text{subject to} & 2x_1 + x_2 - x_3 + x_4 = 1, \\ & -1 \leq x_j \leq 1, \quad j = 1, \ldots, 4. \end{array}$$

Solve (LP) by Dantzig-Wolfe decomposition. Consider $2x_1 + x_x - x_3 + x_4 = 1$ the complicating constraint, and consider $-1 \leq x_j \leq 1$, $j = 1, \ldots, 4$, the easy constraints.

Use the extreme points $v_1 = (-1 \;\; -1 \;\; -1 \;\; -1)^T$ and $v_2 = (1 \;\; -1 \;\; -1 \;\; 1)^T$ for obtaining an initial feasible solution to the master problem.

The subproblem(s) that arise may be solved in any way, that need not be systematic. .... (10p)

5. In the course, we have dealt with stochastic problems by minimizing the expected value. For example, for a set of scenarios $s \in \{1, \ldots, S\}$, with given probabilities p_s, $s = 1, \ldots, S$, a set of variables $x \in X$ and a function $f(x, s)$ defined for each x and s, the corresponding optimization problem takes the form

$$(P_{expected}) \quad \underset{x \in \mathcal{X}}{\text{minimize}} \quad \textstyle\sum_{s=1}^{S} p_s f(x, s).$$

Another measure would be robust optimization, where the worst outcome is minimized, i.e., the optimization problem is written

$$(P_{robust}) \quad \underset{x \in \mathcal{X}}{\text{minimize}} \quad \max_{s=1,\ldots,S} \; f(x, s).$$

Conditional value at risk (CVaR) is a risk measure used when dealing with stochastic problems, for example in optimization of radiation therapy or finance. This measure is parameterized by a nonnegative scalar α. For a fixed α, it is given by the solution to the optimization problem

$$(P_\alpha) \quad \begin{array}{lll} \underset{x \in \mathcal{X}}{\text{minimize}} & \underset{\pi \in I\!R^S}{\text{maximize}} & \sum_{s=1}^{S} \pi_s f(x, s) \\ & \text{subject to} & \sum_{s=1}^{S} \pi_s = 1, \\ & & 0 \leq \pi_s \leq \frac{1}{\alpha} p_s, \quad s = 1, \ldots, S. \end{array}$$

For a fixed x, the inner maximization problem may be interpreted as finding the worst possible probability distribution π in the sense that the expected outcome is maximized, subject to $\pi \leq (1/\alpha)p$.

(a) Show that for $\alpha = 1$, (P_α) is equivalent to $(P_{expected})$. .... (2p)
Hint: In (P_α), for a fixed x, what are the feasible values of π when $\alpha = 1$?

(b) Show that for $\alpha \leq \min_{s=1,\ldots,S} p_s$, (P_α) is equivalent to (P_{robust}). .... (2p)
Hint: In (P_α), for a fixed x, what are the feasible values of π when $\alpha \leq \min_{s=1,\ldots,S} p_s$?

(c) Use your expertise in linear programming to rewrite (P_α) as one minimization problem without the inner maximization. .... (6p)
Hint: Note that for a fixed x in (P_α), the inner maximization problem is a linear program.

Good luck!