SF2812 Applied linear optimization, final exam
Wednesday March 18 2015 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the linear program

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 7 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 & -1 & -2 & 0 \end{pmatrix}^T.$$

Let $\widehat{x} = (2\ 2\ 1\ 1\ 0)^T$.

(a) Show that $\widehat{x}$ is a feasible solution to (LP). In addition, determine two basic feasible solutions to (LP) such that they have the same objective function value as $\widehat{x}$. .... (4p)
Hint: It holds that $Av = 0$ for $v = (1\ \ -1\ \ -1\ \ \ 1\ \ 0)^T$.

(b) Show that $\widehat{x}$ is optimal to (LP). .... (4p)

(c) Let $\widehat{y}$ and $\widehat{s}$ denote optimal solutions to the corresponding dual problem. Motivate why $\widehat{s}_i = 0$, $i = 1, 2, 3, 4$. .... (2p)

2. Consider the optimization problem (P) given by

$$(P) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & \|x\|_1 \leq 1, \end{array}$$

with

$$A = \begin{pmatrix} 3 & 1 & -2 & -1 \end{pmatrix}, \quad b = 2, \quad c = \begin{pmatrix} -4 & 0 & 2 & 2 \end{pmatrix}^T.$$

Your task is to solve (P) using Dantzig-Wolfe decomposition taking into account problem structure. Consider $Ax = b$ the complicating constraints, and let $S =$

$\{x \in I\!R^4 : \|x\|_1 \le 1\}$ denote the easy constraints. Note that $S = \{x \in I\!R^4 : |x|_1 + |x|_2 + |x|_3 + |x|_4 \le 1\}$, so it is a polytope that can be described by linear equality and inequality constraints, and the extreme points of S are $(1\ \ 0\ \ 0\ \ 0)^T$, $(-1\ \ 0\ \ 0\ \ 0)^T$, $(0\ \ 1\ \ 0\ \ 0)^T$, $(0\ \ -1\ \ 0\ \ 0)^T$ etc.

In the master problem, start with the basis that corresponds to the extreme points $(1\ \ 0\ \ 0\ \ 0)^T$ and $(0\ \ -1\ \ 0\ \ 0)^T$. ....(10p)

3. Consider the linear program

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \ge 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & -1 & 2 & 1 \end{pmatrix}^T.$$

Assume that we want to solve (LP) using a primal-dual interior-point method. Assume further that we initially choose $x^{(0)} = (4\,2\,2\,1)^T$, $y^{(0)} = (0\,0)^T$, $s^{(0)} = (1\,2\,2\,4)^T$, and $\mu = 4$. Here, y and s denote the dual variables.

(a) Formulate the linear system of equations to be solved in the first iteration of the primal-dual interior-point method for the given initial values. First formulate the general form and then add explicit numerical values into the system of equations. ....(7p)

(b) The solution to the above system of linear equations is given by

$$\Delta x = \begin{pmatrix} -2.4 \\ 0.4 \\ -3.6 \\ -0.4 \end{pmatrix}, \quad \Delta y = \begin{pmatrix} -0.6 \\ -1.0 \end{pmatrix}, \quad \Delta s = \begin{pmatrix} 0.6 \\ -0.4 \\ 3.6 \\ 1.6 \end{pmatrix}.$$

Show how these values may be used to determine $x^{(1)}$, $y^{(1)}$ and $s^{(1)}$ in a suitable way. Complete the calculations up to the point where you would need a calculator. ....(3p)

4. Consider the integer program (IP) defined by

$$(IP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \ge b, \\ & Cx \ge d, \\ & x \ge 0, \quad x \text{ integer}. \end{array}$$

Let z_{IP} denote the optimal value of (IP).

Associated with (IP) we may define the dual problem (D) as

$$(D) \qquad \begin{array}{ll} \text{maximize} & \varphi(u) \\ \text{subject to} & u \geq 0, \end{array}$$

where $\varphi(u) = \min\{c^T x + u^T(b - Ax) : Cx \geq d, x \geq 0 \text{ integer}\}$. Let z_D denote the optimal value of (D).

Let (LP) denote the linear program obtained from (IP) by relaxing the integer requirement, i.e.,

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \\ & Cx \geq d, \\ & x \geq 0. \end{array}$$

Let z_{LP} denote the optimal value of (LP).

Show that $z_{IP} \geq z_D \geq z_{LP}$. ....(10p)

5. A not so reliable person named AF has been asked to solve the linear program

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^T.$$

AF has solved the problem using the simplex method and come up with $\widetilde{x} = (1\ 2\ 0\ 0)^T$, $\widetilde{y} = (1\ {-1})^T$ and $\widetilde{s} = (0\ 0\ 2\ 2)^T$ as optimal solutions to (LP) and its corresponding dual.

He then realizes that he has made a mistake and solved the problem with $b_1 = 3$ instead of the correct value $b_1 = 1$. He now turns to you for assistance.

Help AF to solve the problem using the dual simplex method. Take the basis provided by AF in his solution as your initial basis. Also motivate for AF why the dual simplex method would be a good choice in this situation. ....(10p)

Good luck!