1. Consider a transportation problem \((TP)\) defined as

\[
\text{minimize} \quad \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} \\
\text{subject to} \quad \sum_{j=1}^{4} x_{ij} = a_i, \quad i = 1, 2, 3, \\
\sum_{i=1}^{3} x_{ij} = b_j, \quad j = 1, 2, 3, 4, \\
x_{ij} \geq 0, \quad i = 1, 2, 3, \quad j = 1, 2, 3, 4,
\]

where

\[
C = \begin{pmatrix} 4 & 2 & 5 & 1 \\ 7 & 4 & 7 & 5 \\ 6 & 4 & 6 & 2 \end{pmatrix}, \quad a = \begin{pmatrix} 8 \\ 12 \\ 10 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 8 \\ 7 \\ 9 \end{pmatrix}.
\]

The dual problem associated with \((TP)\) may be written as

\[
\text{maximize} \quad \sum_{i=1}^{3} a_i u_i + \sum_{j=1}^{4} b_j v_j \\
\text{subject to} \quad u_i + v_j \leq c_{ij}, \quad i = 1, 2, 3, \quad j = 1, 2, 3, 4.
\]

You have been given \(\hat{X}, \hat{u}\) and \(\hat{v}\) as

\[
\hat{X} = \begin{pmatrix} 6 & 1.5 & 0 & 0.5 \\ 0 & 6.5 & 5.5 & 0 \\ 0 & 0 & 1.5 & 8.5 \end{pmatrix}, \quad \hat{u} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \hat{v} = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 0 \end{pmatrix}.
\]
(a) A friend of yours, who has not taken this course, claims that \( \hat{X} \) cannot be optimal to \((TP)\), since the transportation problem should have integer valued optimal solutions when \( a \) and \( b \) are integers. Comment on your friend’s claim.

(b) Verify that \( \hat{X} \) is optimal to \((TP)\) and that \( \hat{u}, \hat{v} \) is optimal to \((DTP)\). .. . (3p)

Hint: With \( S \) given by
\[
S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 \end{pmatrix},
\]

(c) Find, using \( \hat{X} \), two integer valued optimal solutions to \((TP)\). ............... (3p)

(d) Explain why you would not obtain \( \hat{X} \) as an answer if you used the simplex method to solve \((TP)\). .................. (2p)

2. Consider fitting a line \( y = kx + l \) to a number of given points \((x_i, y_i), i = 1, \ldots, m\). In particular, \( k \) and \( l \) should be chosen so that the maximum deviation in the \( y \)-direction is minimized, i.e., \( k \) and \( l \) are chosen according to \( \min_{k,l} \max_{i=1,\ldots,m} |kx_i + l - y_i| \). By introducing the extra variable \( z \), the problem may be written as an LP problem on the form

\[
\text{(LP)} \quad \begin{array}{l}
\text{minimize} & z \\
\text{subject to} & -z \leq kx_i + l - y_i \leq z, \quad i = 1, \ldots, m,
\end{array}
\]

where \( x_i, i = 1, \ldots, m \) and \( y_i, i = 1, \ldots, m \), are given parameters, and \( k, l \) and \( z \) are the variables. We assume that \( m \geq 3 \) and \( x_i \neq x_j \) for \( i \neq j \).

(a) Formulate the dual problem \((DLP)\) associated with \((LP)\). ............... (5p)

(b) Given an optimal solution \( k, l, z \) to this line fitting problem, show that there are at least three points among the given points \((x_i, y_i), i = 1, \ldots, m\), at which \( |kx_i + l - y_i| = z \). You should motivate this result based on properties of the linear programs. .................. (5p)

3. Let \((P)\) and \((D)\) be defined by

\[
(P) \quad \begin{array}{l}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b, \\
& x \geq 0,
\end{array}
\]

and

\[
(D) \quad \begin{array}{l}
\text{maximize} & b^T y \\
\text{subject to} & A^T y + s = c, \\
& s \geq 0.
\end{array}
\]
For a fixed positive barrier parameter $\mu$, consider the primal-dual nonlinear equations

\[
\begin{align*}
Ax &= b, \\
A^T y + s &= c, \\
XSe &= \mu e,
\end{align*}
\]

where we in addition require $x > 0$ and $s > 0$. Here, $X = \text{diag}(x)$, $S = \text{diag}(s)$ and $e$ is an $n$-vector with all components one.

(a) Assume that $x(\mu)$, $y(\mu)$ and $s(\mu)$ solve the primal-dual nonlinear equations for a given positive $\mu$, with $x(\mu) > 0$ and $s(\mu) > 0$. Show that $x(\mu)$ is feasible to $(P)$ and $y(\mu), s(\mu)$ are feasible to $(D)$ with duality gap $\eta\mu$. .................(3p)

(b) Derive the system of linear equations that results when the primal-dual nonlinear equations are solved by Newton’s method. .........................(5p)

(c) How are the implicit constraints $x > 0$ and $s > 0$ handled in a Newton-based interior method that approximately solves the primal-dual system of nonlinear equations for a sequence of decreasing values of $\mu$? .........................(2p)

4. Consider the binary integer programming problem $(IP)$ given by

\[
\begin{align*}
\text{minimize} & \quad -5x_1 - 7x_2 - 10x_3 \\
\text{subject to} & \quad -3x_1 - 6x_2 - 7x_3 \geq -8, \\
& \quad -x_1 - 2x_2 - 3x_3 \geq -3, \\
& \quad x_j \in \{0, 1\}, \quad j = 1, \ldots, n.
\end{align*}
\]

Assume that the constraint $-3x_1 - 6x_2 - 7x_3 \geq -8$ is relaxed by Lagrangian relaxation for a nonnegative multiplier $u$.

(a) For $u = 1$, compute two optimal solutions to the resulting Lagrangian relaxed problem. The Lagrangian relaxed problem may be solved by any method, that need not be systematic. .........................(4p)

(b) Use the two optimal solutions to the Lagrangian relaxed problem computed in Exercise 4a to give two different subgradients to the dual objective function $\varphi$ at $u = 1$. .........................(4p)

(c) Show that there is a convex combination of the two subgradients computed in Exercise 4b that gives the zero vector. What is the implication for the dual problem? .........................(2p)

5. Consider a cutting-stock problem with the following data:

\[
W = 11, \quad m = 3, \quad w_1 = 3, \quad w_2 = 5, \quad w_3 = 9, \quad b = \begin{pmatrix} 60 & 50 & 40 \end{pmatrix}^T.
\]

Notation and problem statement are in accordance to the textbook. Given are rolls of width $W$. Rolls of $m$ different widths are demanded. Roll $i$ has width $w_i$, $i = 1, \ldots, m$. The demand for roll $i$ is given by $b_i$, $i = 1, \ldots, m$. The aim is to cut the $W$-rolls so that a minimum number of $W$-rolls are used.
(a) Solve the the LP-relaxed problem associated with the above problem. Start
with the basic feasible solution associated with the three “pure” cut patterns
$(3 \; 0 \; 0)^T$, $(0 \; 2 \; 0)^T$ and $(0 \; 0 \; 1)^T$. The subproblems that arise may be solved in
any way, that need not be systematic. ................................................. (8p)

(b) Determine a “near-optimal” solution to the original problem. Give a bound on
the maximum deviation from the optimal value of the original problem. . . (2p)

Good luck!